On multi-graviton and multi-gravitino gauge theories

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Abstract

This paper studies nonlinear deformations of the linear gauge theory of any number of spin-2 and spin-3/2 fields with general formal multiplication rules in place of standard Grassmann rules for manipulating the fields, in four spacetime dimensions. General possibilities for multiplication rules and coupling constants are simultaneously accommodated by regarding the set of fields equivalently as a single algebra-valued spin-2 field and single algebra-valued spin-3/2 field, where the underlying algebra is factorized into a field-coupling part and an internal multiplication part. The condition that there exist a gauge invariant Lagrangian (to within a divergence) for these algebra-valued fields is used to derive determining equations whose solutions give all allowed deformation terms, yielding nonlinear field equations and nonabelian gauge symmetries, together with all allowed formal multiplication rules as needed in the Lagrangian for demonstration of invariance under the gauge symmetries and for derivation of the field equations. In the case of spin-2 fields alone, the main result of this analysis is that all deformations (without any higher derivatives than appear in the linear theory) are equivalent to an algebra-valued Einstein gravity theory. By a systematic examination of factorizations of the algebra, a novel type of nonlinear gauge theory of two or more spin-2 fields is found, where the coupling for the fields is based on structure constants of an anticommutative, anti-associative algebra, and with formal multiplication rules that make the fields anticommuting (while products obey anti-associativity). Supersymmetric extensions of these results are obtained in the more general case when spin-3/2 fields are included.

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I. INTRODUCTION AND SUMMARY

It has long been known [1, 2, 3, 4] that the Einstein gravitational field equations describe a nonlinear gauge theory of a massless spin-2 field (i.e. graviton) as defined by the difference of the gravitational metric tensor and any fixed background flat metric, where the gauge symmetry arises from diffeomorphisms on the metric field tensor. Moreover, the uniqueness of this theory in providing a consistent nonlinear self-coupling for a spin-2 field is by now well established from many points of view [5, 6, 7, 8, 9, 10, 11]. Nevertheless, there has been some interest in recent years in the possibility of consistent novel nonlinear gauge theories of spin-2 fields. This paper significantly elaborates one avenue of work on such possibilities.

The motivation is explained by certain features of classical supergravity theories [12, 13, 14], viewed as a supersymmetric extension of the Einstein gravity theory involving, as a matter source, a massless spin-3/2 field (i.e. gravitino) that is required to be formally anticommuting. In this extension, the massless spin-2 field remains formally commuting, while products of the spin-2 and spin-3/2 fields are manipulated by formal rules of Grassmann multiplication. These rules, which serve as classical counterparts of graviton-gravitino quantum field commutation relations, are used for manipulations in the Lagrangian to establish supersymmetry invariance and to derive the field equations. N= 1 supersymmetry involves a single pair of spin-2 and spin-3/2 fields, but for N> 1 a complete supersymmetry multiplet [14] requires more fields, and notably, additional spin-2 fields are needed if N> 8.

These features naturally suggest exploring the possibilities for nonlinear gauge theories of, firstly, a set of at least two ordinary (commuting) spin-2 fields, and secondly, a single spin-2 field with formal multiplication rules that make it noncommuting. Mathematically, note that a set of $n \geq 1$ ordinary spin-2 fields is equivalent to a single spin-2 field possessing an internal structure of a real n-dimensional vector space. Consequently, in a nonlinear theory the coupling constants that appear in the Lagrangian for a set of ordinary spin-2 fields thereby correspond to an algebraic structure on the internal vector space possessed by an equivalent single algebra-valued spin-2 field. The resulting algebraic structure, furthermore, serves to provide multiplication rules for manipulating products of the algebra-valued spin-2 field in the Lagrangian. This discussion shows that there is a well-defined mathematical equivalence between nonlinear theories of a set of ordinary (i.e. real-valued) spin-2 fields and nonlinear theories of a single spin-2 field with formal multiplication rules represented

by an internal algebra in which the spin-2 field takes values. In particular, under this point of view, Einstein gravity theory for a single commuting spin-2 field formulated using even-Grassmann multiplication rules is the same as a nonlinear theory found by Cutler and Wald [15, 16] for a set of real-valued spin-2 fields with coupling constants corresponding to any even-Grassmann algebra. More precisely, the equivalence here is such that each spin-2 field is associated with a basis element in this algebra [16].

These considerations carry over in an obvious way to spin-3/2 (and other) fields. As a result, N= 1 supergravity theory for a single commuting spin-2 field and a single anticommuting spin-3/2 field based on Grassmann multiplication rules has an equivalent formulation as a nonlinear theory of a set of ordinary (real-valued) spin-2 and spin-3/2 fields whose coupling constants are associated with any Grassmann algebra [17, 18]. This motivates a fundamental question of whether classical N= 1 supergravity theory formulated in terms of a Grassmann algebra is the unique possibility for a nonlinear gauge theory of a set of ordinary spin-2 and spin-3/2 fields; and likewise, whether its graviton sector given by the formulation of Einstein gravity theory involving an even-Grassmann algebra is the unique possibility for a nonlinear gauge theory of a set of ordinary spin-2 fields alone.

The most systematic approach for determining all possibilities for nonlinear spin-2/spin-3/2 gauge theories is by a deformation analysis of linear abelian gauge theory of a set of ordinary spin-2 and spin-3/2 fields. Here, deformations refer to adding quadratic and higher power terms in the linear field equations while adding linear and higher power terms in the abelian gauge symmetries, such that there exists a gauge invariant Lagrangian (to within a total divergence), with the undeformed Lagrangian of the linear abelian theory not being equivalent to the deformed Lagrangian under field redefinitions. The condition of gauge invariance can be used to obtain determining equations to solve for the allowed form of the deformation terms order by order in powers of the fields (see Refs. [19, 20]). Two natural restrictions on the general form considered for these terms come from requiring that the deformations preserve the number of gauge degrees of freedom and initial-data degrees of freedom for the fields. This leads to restricting any derivatives in the deformed gauge symmetries and field equations to be of no higher order than those in the linear theory [18], which will then be referred to as a non-higher-derivative deformation.

A complete analysis of non-higher-derivative deformations for a set of arbitrarily many spin-2 and spin-3/2 fields in four spacetime dimensions, without further restrictions or special

assumptions on possible forms for gauge symmetries and field equations, was first carried out in Ref. [18] using a field theoretic formulation of the deformation determining equations. This analysis obtained two strong uniqueness results: First, for a set of spin-2 fields alone, the only non-higher-derivative deformations are equivalent to Einstein gravity theory for an even-Grassmann algebra-valued spin-2 field. However, for a set of spin-2 and spin-3/2 fields, the allowed non-higher-derivative deformations correspond to a chiral generalization of classical N= 1 supergravity theory for an algebra-valued pair of spin-2 and spin-3/2 fields involving a novel modification of a Grassmann algebra such that the spin-3/2 field is only anticommutative in combination with charge conjugation, while the spin-2 field remains commutative, but commutativity (and associativity) of the spin-2 field in products with the spin-3/2 field holds only in combination with charge conjugation. The origin of this generalization stems from the Weyl-spinor formulation of supergravity theory, in which the anticommuting nature of the spin-3/2 field is found to be never needed separately from charge conjugation [21]. This allows dispensing with certain Grassmann multiplication rules, and thereby defines a non-Grassmann multiplication intertwined with charge conjugation. As a consequence, the supergravity field equations and supersymmetry become deformed by some chiral terms that otherwise would vanish if the spin-3/2 field were strictly anticommuting. These terms essentially maintain the gauge invariance for the deformation. The formulation of the resulting chiral generalized N= 1 supergravity Lagrangian [21] in terms of a pair of non-Grassmann algebra-valued spin-2 and spin-3/2 fields is explained in Ref. [18].

It is worth emphasizing that, due to gauge invariance, the coupling of the spin-3/2 field to the spin-2 field in this generalized N= 1 supergravity theory is completely consistent. Moreover, when viewed as a nonlinear gauge theory of a pair of spin-2 and spin-3/2 fields with formal internal multiplication rules, it shares the same key features as standard classical N= 1 supergravity theory — well-posedness of the initial value problem [23], formal positive energy properties [24], and a geometrical description [25] in terms of curvature of the metric tensor associated to the spin-2 field, with a matter source and torsion determined by the spin-3/2 field. In the corresponding formulation for a set of ordinary spin-2 and spin-3/2 fields, the field equations continue to be well-posed and retain a geometrical meaning in terms of an algebra-valued metric and curvature given by the spin-2 fields, as discussed in detail in Ref. [16], and an algebra-valued torsion and matter source determined by the spin-3/2 fields, outlined in Ref. [18]. However, the field equations viewed in this manner

have a partially decoupled nonlinear structure, where the coupling terms reflect the algebra multiplication relations among the basis elements of the internal algebra [17]. As a result of this structure, the canonical stress-energy tensor obtained from the Lagrangian for the set of field equations is found to yield a total energy that, in general, is of indefinite sign. (This feature has been studied in Ref. [22] for an analogous nonlinear theory of a set of scalar fields, describing a standard quartic self-coupling for an equivalent algebra-valued single scalar field, with a simple choice of internal algebra.)

The non-positivity of energy for these coupled ordinary spin-2 and spin-3/2 fields is directly related to the multiplication being nontrivial in the internal algebra on which the nonlinear gauge theory is based. Indeed, in Ref. [26] it was subsequently shown that in the case of a set of spin-2 fields the only internal algebra yielding a nonlinear gauge theory whose canonical energy is positive is given by a direct sum of one-dimensional unit algebras. (This feature extends to the more general case of a set of spin-2 and spin-3/2 fields [27].) Ref. [26] also gave a deformation analysis that generalized the uniqueness result for even-Grassmann valued Einstein gravity theory as a nonlinear gauge theory for a set of ordinary spin-2 fields, by relaxing the natural restriction on highest order derivatives in deformations of the spin-2 gauge symmetries (and allowing other than four spacetime dimensions) through the use of powerful BRST cohomology techniques [28] to formulate and solve the deformation determining equations.

To-date, all previous investigations of nonlinear gauge theories for a set of spin-2 fields, and more generally, a set of spin-2 and spin-3/2 fields, have considered only ordinary (i.e. real-valued) fields. This excludes, consequently, the possibility of more than one anticommuting spin-3/2 field with formal odd-Grassmann multiplication rules, as would arise in N$>$ 1 supersymmetry multiplets. It also excludes the more exotic possibility of a noncommuting spin-2 field. The main purpose of the present paper is to fill the previous gaps by giving a systematic determination of all possible nonlinear gauge theories of an arbitrary number of spin-2 and spin-3/2 fields each with its own internal formal multiplication rules, and as a special case, spin-2 fields with other than even-Grassmann multiplication rules, in four spacetime dimensions. The results yield an exotic classical nonlinear gauge theory of anticommuting spin-2 fields, and a supersymmetric extension with commuting spin-3/2 fields, employing formal multiplication rules in which the usual spin-statistics relation for the spin-2/spin-3/2 fields is reversed at the classical level. (These nonlinear theories turn

out to involve only products of distinct fields, but never a product of any field with itself, so anticommutativity is not needed to hold for each individual spin-2 field and likewise for commutativity of each individual spin-3/2 field, in remarkable accordance with the general spin-statistics relations [29] allowed by quantum field theory.) In Sec. II the deformation analysis used to find these theories is summarized, and uniqueness results from this analysis are stated. The theories are presented in detail in Sec. III. Some features of the theories are discussed along with a few concluding remarks in Sec. IV. An appendix summarizes some material on relevant algebras.

II. DEFORMATION ANALYSIS

We begin by setting up the framework for deformations, using a generalization of the formalism (and notation) of Ref. [18] to accommodate any number of spin-2 and spin-3/2 fields with an internal vector space structure for any formal multiplication rules.

A. Preliminaries

The spin-2 fields are taken to be real spinorial tensors ${h_{aBB'}}^{\mu}$, $\mu = 1, \ldots, n$, and the spin-3/2 fields are taken to be complex vector-spinors ${\psi_{aB}}^{\Lambda}$, $\Lambda = 1, \ldots, n'$. This choice of variables is motivated by the chiral spinorial formulation of classical N= 1 supergravity theory [30] that uses a null spinorial tetrad ${e^{BB'}}_{a}$ and a Weyl vector-spinor φ_a^B which are, respectively, even and odd Grassmann-valued. Linearization about a flat tetrad $\sigma_{aBB'}$ and a zero vector-spinor in that theory yields a spin-2 field $h_{aBB'} = e_{aBB'} - \sigma_{aBB'}$ and a spin-3/2 field $\psi_{aB} = \varphi_{aB}$, where $n = n' = 1$.

Each field ${h_{aBB'}}^{\mu}, {\psi_{aB}}^{\Lambda}$ here is regarded as taking values in an internal vector space $\mathbf{X}, \mathbf{Y}$, where we fix $\mathbf{X}$ to be a real vector space of some arbitrary dimension with a basis $\mathbf{x}_1, \mathbf{x}_2, \ldots$, and $\mathbf{Y}$ to be a complexified vector space of some arbitrary dimension with a basis $\mathbf{y}_1, \mathbf{y}_2, \ldots$, respectively. The structure necessary to formulate internal multiplication rules for ${h_{aBB'}}^{\mu}$ and ${\psi_{aB}}^{\Lambda}$ will be given by multilinear maps from products of $\mathbf{X}, \mathbf{Y}$, into $\mathbf{X}$ and $\mathbf{Y}$. Note that the coefficients in a basis expansion of such products define multiplication structure constants, which represent the multiplication rules. With respect to these bases, we expand

the fields

$$h_{aBB'}{}^{\mu} = h_{aBB'}{}^{\mu,1}\mathbf{x}_1 + h_{aBB'}{}^{\mu,2}\mathbf{x}_2 + \cdots, \quad \psi_{aB}{}^{\Lambda} = \psi_{aB}{}^{\Lambda,1}\mathbf{y}_1 + \psi_{aB}{}^{\Lambda,2}\mathbf{y}_2 + \cdots \tag{2.1}$$

where $h_{aBB'}{}^{\mu,1}, h_{aBB'}{}^{\mu,2}, \ldots, \psi_{aB}{}^{\Lambda,1}, \psi_{aB}{}^{\Lambda,2}, \ldots$ are ordinary real-valued spinorial tensor fields and complex-valued vector-spinor fields. Products of $h_{aBB'}{}^{\mu}$ and $\psi_{aB}{}^{\Lambda}$ involving internal multiplication rules then reduce to ordinary products of $h_{aBB'}{}^{\mu,1}, h_{aBB'}{}^{\mu,2}, \ldots$ and $\psi_{aB}{}^{\Lambda,1}, \psi_{aB}{}^{\Lambda,2}, \ldots$ as specified by the multiplication structure constants. This allows the set of ordinary spin-2 and and spin-3/2 fields $h_{aBB'}{}^{\mu,\mathcal{A}}, \psi_{aB}{}^{\Lambda,\mathcal{A}'}$ ($\mathcal{A} = 1, 2, \ldots, \mathcal{A}' = 1, 2, \ldots$) to be used as the field variables for the subsequent framework here. For this purpose it is convenient hereafter to employ the multi-index notation $\mathcal{A}_\mu = (\mu, \mathcal{A})$, $\mathcal{A}'_\Lambda = (\Lambda, \mathcal{A}')$. The summation convention with respect to a repeated index $\mathcal{A}_\mu$ will mean a sum over both μ and $\mathcal{A}$, and similarly, a sum over both Λ and $\mathcal{A}'$ for a repeated index $\mathcal{A}'_\Lambda$.

To proceed, we start from the linear abelian gauge theory for the set of $n \geq 1$ spin-2 fields and $n' \geq 1$ spin-3/2 fields each with an internal vector space structure (2.1), on a flat 4-dimensional spacetime manifold. In terms of the ordinary field variables $h_{aBB'}{}^{\mathcal{A}_\mu}, \psi_{aB}{}^{\mathcal{A}'_\Lambda}$, the linear spin-2 field equations are given by the Fierz-Pauli equation

$$0 = \partial^c\partial_c\gamma_{ab}{}^{\mathcal{A}_\mu} - 2\partial^c\partial_{(a}\gamma_{b)c}{}^{\mathcal{A}_\mu} + \eta_{ab}\partial^c\partial^d\gamma_{cd}{}^{\mathcal{A}_\mu} = 4\overset{(1)}{E}{}_{h\,ab}^{\mathcal{A}_\mu} \tag{2.2}$$

with $\gamma_{ab}{}^{\mathcal{A}_\mu} = h_{ab}{}^{\mathcal{A}_\mu} - \frac{1}{2}\eta_{ab}h_c{}^{c\mathcal{A}_\mu}$ where $h_{ab}{}^{\mathcal{A}_\mu} = \sigma_{(b}{}^{BB'}h_{a)BB'}{}^{\mathcal{A}_\mu}$, and the linear spin-3/2 field equations are given by the Rarita-Schwinger equation

$$0 = \epsilon_a{}^{bcd}\sigma_{bB}{}^{B'}\partial_c\bar{\psi}_{dB'}{}^{\mathcal{A}'_\Lambda} = 2i\overset{(1)}{E}{}_{\psi\,aB}^{\mathcal{A}'_\Lambda} \tag{2.3}$$

where $\sigma_{bBB'}$ is any flat spinorial tetrad and $\epsilon_{abcd} = 2i\sigma_{[a|A'|}{}^{A}\sigma_{b|B|}{}^{A'}\sigma_{c|B'|}{}^{B}\sigma_{d]A}{}^{B'}$ is the associated volume tensor of the flat metric $\eta_{ab} = \sigma_a{}^{CC'}\sigma_{bCC'}$. The abelian gauge symmetries on these field variables consist of infinitesimal variations given by a linearized general-covariance symmetry

$$\overset{(0)}{\delta}_\xi h_{aBB'}{}^{\mathcal{A}_\mu} = 2\sigma^b{}_{BB'}\partial_{(a}\xi_{b)}{}^{\mathcal{A}_\mu}, \quad \overset{(0)}{\delta}_\xi\psi_{aB}{}^{\mathcal{A}'_\Lambda} = 0, \tag{2.4}$$

a linearized supersymmetry

$$\overset{(0)}{\delta}_\zeta\psi_{aB}{}^{\mathcal{A}'_\Lambda} = \partial_a\zeta_B{}^{\mathcal{A}'_\Lambda}, \quad \overset{(0)}{\delta}_\zeta h_{aBB'}{}^{\mathcal{A}_\mu} = 0, \tag{2.5}$$

and a linearized local Lorentz symmetry

$$\overset{(0)}{\delta}_\chi h_{aBB'}{}^{\mathcal{A}_\mu} = \sigma_{aB'}{}^{A}\chi_{AB}{}^{\mathcal{A}_\mu} + c.c., \quad \overset{(0)}{\delta}_\zeta\psi_{aB}{}^{\mathcal{A}'_\Lambda} = 0, \tag{2.6}$$

which involve as respective variables the arbitrary covector fields $\xi_a{}^{\mathcal{A}_\mu}$, spinor fields $\zeta_B{}^{\mathcal{A}'_\Lambda}$, and symmetric spinor fields $\chi_{AB}{}^{\mathcal{A}_\mu} = \chi_{(AB)}{}^{\mathcal{A}_\mu}$. The field equations have a gauge invariant Lagrangian formulation given by

$$\overset{(2)}{L} = \tfrac{1}{2} q_{\mathcal{A}_\mu \mathcal{B}_\nu} h_{aBB'}{}^{\mathcal{A}_\mu} \overset{(1)}{E}_h{}^{aBB'\mathcal{B}_\nu} + \tfrac{1}{2}(q'{}_{\mathcal{A}'_\Lambda \mathcal{B}'_\Gamma} \bar{\psi}_{aB'}{}^{\mathcal{A}'_\Lambda} \overset{(1)}{\bar{E}}_\psi{}^{aB'\mathcal{B}'_\Gamma} + c.c.) \tag{2.7}$$

where $q_{\mathcal{A}_\mu \mathcal{B}_\nu}$ and $q'{}_{\mathcal{A}'_\Lambda \mathcal{B}'_\Gamma}$ are, respectively, components of any fixed diagonal real-symmetric and skew-hermitian nondegenerate matrices. In particular, through Euler-Lagrange operators $\mathrm{E}_{h_{aBB'}}^{\mathcal{A}_\mu}(\,\cdot\,)$ and $\mathrm{E}_{\psi_{aB}}^{\mathcal{A}'_\Lambda}(\,\cdot\,)$, which annihilate total divergences, the Lagrangian (2.7) yields the field equations

$$\overset{(1)}{E}{}_{h_{ab}}^{\mathcal{A}_\mu} = \mathrm{E}_{h_{ab}}^{\mathcal{A}_\mu}(\overset{(2)}{L}), \quad \overset{(1)}{E}{}_{\psi_{aB}}^{\mathcal{A}'_\Lambda} = \mathrm{E}_{\psi_{aB}}^{\mathcal{A}'_\Lambda}(\overset{(2)}{L}), \tag{2.8}$$

while invariance with respect to the gauge symmetries is expressed by

$$\mathrm{E}_{h_{aBB'}}^{\mathcal{A}_\mu}(\overset{(0)}{\delta}_\xi \overset{(2)}{L}) = \mathrm{E}_{h_{aBB'}}^{\mathcal{A}_\mu}(\overset{(0)}{\delta}_\zeta \overset{(2)}{L}) = \mathrm{E}_{h_{aBB'}}^{\mathcal{A}_\mu}(\overset{(0)}{\delta}_\chi \overset{(2)}{L}) = 0, \tag{2.9}$$

$$\mathrm{E}_{\psi_{aB}}^{\mathcal{A}'_\Lambda}(\overset{(0)}{\delta}_\xi \overset{(2)}{L}) = \mathrm{E}_{\psi_{aB}}^{\mathcal{A}'_\Lambda}(\overset{(0)}{\delta}_\zeta \overset{(2)}{L}) = \mathrm{E}_{\psi_{aB}}^{\mathcal{A}'_\Lambda}(\overset{(0)}{\delta}_\chi \overset{(2)}{L}) = 0. \tag{2.10}$$

A deformation of this linear theory is defined by adding terms of linear and higher powers to the abelian gauge symmetries

$$\overset{(0)}{\delta} h_{aBB'}{}^{\mathcal{A}_\mu} + \overset{(1)}{\delta} h_{aBB'}{}^{\mathcal{A}_\mu} + \cdots = \delta h_{aBB'}{}^{\mathcal{A}_\mu}, \quad \overset{(0)}{\delta} \psi_{aB}{}^{\mathcal{A}'_\Lambda} + \overset{(1)}{\delta} \psi_{aB}{}^{\mathcal{A}'_\Lambda} + \cdots = \delta \psi_{aB}{}^{\mathcal{A}'_\Lambda}, \tag{2.11}$$

while simultaneously adding terms of quadratic and higher powers to the linear field equations

$$\overset{(1)}{E}{}_{h_{aBB'}}^{\mathcal{A}_\mu} + \overset{(2)}{E}{}_{h_{aBB'}}^{\mathcal{A}_\mu} + \cdots = E_{h_{aBB'}}^{\mathcal{A}_\mu}, \quad \overset{(1)}{E}{}_{\psi_{aB}}^{\mathcal{A}'_\Lambda} + \overset{(2)}{E}{}_{\psi_{aB}}^{\mathcal{A}'_\Lambda} + \cdots = E_{\psi_{aB}}^{\mathcal{A}'_\Lambda}, \tag{2.12}$$

such that there exists a gauge invariant Lagrangian (to within a total divergence)

$$\overset{(2)}{L} + \overset{(3)}{L} + \cdots = L \tag{2.13}$$

satisfying

$$\mathrm{E}_{h_{aBB'}}^{\mathcal{A}_\mu}(\delta L) = 0, \quad \mathrm{E}_{\psi_{aB}}^{\mathcal{A}'_\Lambda}(\delta L) = 0, \tag{2.14}$$

where the Lagrangian is related to the field equations through $\mathrm{E}_{h_{aBB'}}^{\mathcal{A}_\mu}(L) = E_{h_{aBB'}}^{\mathcal{A}_\mu}$ and $\mathrm{E}_{\psi_{aB}}^{\mathcal{A}'_\Lambda}(L) = E_{\psi_{aB}}^{\mathcal{A}'_\Lambda}$. We restrict attention hereafter to non-higher-derivative deformations whose terms are locally constructed from $h_{aBB'}{}^{\mathcal{A}_\mu}$, $\psi_{aB}{}^{\mathcal{A}'_\Lambda}$, and their derivatives, (in addition to the spacetime coordinates) such that at most one derivative in total of $h_{aBB'}{}^{\mathcal{A}_\mu}$, $\psi_{aB}{}^{\mathcal{A}'_\Lambda}$,

${\xi_a}^{\mathcal{A}_\mu}$, ${\zeta_B}^{\mathcal{A}'_\Lambda}$, ${\chi_{AB}}^{\mathcal{A}_\mu}$ appears in the deformed gauge symmetries, and at most two derivatives in total of ${h_{aBB'}}^{\mathcal{A}_\mu}, {\psi_{aB}}^{\mathcal{A}'_\Lambda}$ appear in the deformed field equations. Moreover, any such deformations that are related by invertible, nonlinear locally constructed field redefinitions (i.e. change of variables) of ${h_{aBB'}}^{\mathcal{A}_\mu}$, ${\psi_{aB}}^{\mathcal{A}'_\Lambda}$, ${\xi_a}^{\mathcal{A}_\mu}$, ${\zeta_B}^{\mathcal{A}'_\Lambda}$, ${\chi_{AB}}^{\mathcal{A}_\mu}$ are considered to be equivalent. The condition of local gauge invariance (2.14) provides the determining equations for allowed deformations.

The determining equations have a more useful and geometrical formulation as the following Lie derivative equations. We introduce the Lie derivative operator $\mathcal{L}_\delta$ with respect to field variations $(\delta {h_{aBB'}}^{\mathcal{A}_\mu}, \delta {\psi_{aB}}^{\mathcal{A}'_\Lambda})$ acting on field equations $(E_{h_{aBB'}}^{\mathcal{A}_\mu}, E_{\psi_{aB}}^{\mathcal{A}'_\Lambda})$ by

$$
\begin{aligned}
(\mathcal{L}_\delta E)_{h^{\mathcal{A}_\mu}}^{aBB'} = {} & \delta E_{h^{\mathcal{A}_\mu}}^{aBB'} + E_{h^{\mathcal{B}_\nu}}^{cDD'} \partial \delta {h_{cDD'}}^{\mathcal{B}_\nu} / \partial {h_{aBB'}}^{\mathcal{A}_\mu} + E_{\psi^{\mathcal{B}'_\Gamma}}^{cD} \partial \delta {\psi_{cD}}^{\mathcal{B}'_\Gamma} / \partial {h_{aBB'}}^{\mathcal{A}_\mu} + c.c. \\
& - \partial_e \Big(E_{h^{\mathcal{B}_\nu}}^{cDD'} \partial \delta {h_{cDD'}}^{\mathcal{B}_\nu} / \partial (\partial_e {h_{aBB'}}^{\mathcal{A}_\mu}) + E_{\psi^{\mathcal{B}'_\Gamma}}^{cD} \partial \delta {\psi_{cD}}^{\mathcal{B}'_\Gamma} / \partial (\partial_e {h_{aBB'}}^{\mathcal{A}_\mu}) + c.c. \Big)
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
(\mathcal{L}_\delta E)_{\psi^{\mathcal{A}'_\Lambda}}^{aB} = {} & \delta E_{\psi^{\mathcal{A}'_\Lambda}}^{aB} + E_{\psi^{\mathcal{B}'_\Gamma}}^{cD} \partial \delta {\psi_{cD}}^{\mathcal{B}'_\Gamma} / \partial {\psi_{aB}}^{\mathcal{A}'_\Lambda} + \bar{E}_{\psi^{\mathcal{B}'_\Gamma}}^{cD'} \partial \delta {\bar{\psi}_{cD'}}^{\mathcal{B}'_\Gamma} / \partial {\psi_{aB}}^{\mathcal{A}'_\Lambda} \\
& + E_{h^{\mathcal{B}_\nu}}^{cDD'} \partial \delta {h_{cDD'}}^{\mathcal{B}_\nu} / \partial {\psi_{aB}}^{\mathcal{A}'_\Lambda} - \partial_e \Big(E_{h^{\mathcal{B}_\nu}}^{cDD'} \partial \delta {h_{cDD'}}^{\mathcal{B}_\nu} / \partial (\partial_e {\psi_{aB}}^{\mathcal{A}'_\Lambda}) \\
& + E_{\psi^{\mathcal{B}'_\Gamma}}^{cD} \partial \delta {\psi_{cD}}^{\mathcal{B}'_\Gamma} / \partial (\partial_e {\psi_{aB}}^{\mathcal{A}'_\Lambda}) + \bar{E}_{\psi^{\mathcal{B}'_\Gamma}}^{cD'} \partial \delta {\bar{\psi}_{cD'}}^{\mathcal{B}'_\Gamma} / \partial (\partial_e {\psi_{aB}}^{\mathcal{A}'_\Lambda}) \Big)
\end{aligned}
\tag{2.16}
$$

where

$$
\begin{aligned}
\delta = {} & \delta {h_{cDD'}}^{\mathcal{B}_\nu} \partial / \partial {h_{cDD'}}^{\mathcal{B}_\nu} + \delta {\psi_{cD}}^{\mathcal{B}'_\Gamma} \partial / \partial {\psi_{cD}}^{\mathcal{B}'_\Gamma} + c.c. \\
& + (\partial_e \delta {h_{cDD'}}^{\mathcal{B}_\nu}) \partial / \partial (\partial_e {h_{cDD'}}^{\mathcal{B}_\nu}) + (\partial_e \delta {\psi_{cD}}^{\mathcal{B}'_\Gamma}) \partial / \partial (\partial_e {\psi_{cD}}^{\mathcal{B}'_\Gamma}) + c.c.
\end{aligned}
\tag{2.17}
$$

defines the field variation operator. Here, note, we have taken into account the restrictions on highest orders of derivatives in the field equations and field variations as relevant for non-higher-derivative deformations.

Proposition 1. *Local gauge invariance holds iff the Lie derivative of the field equations with respect to the gauge symmetries vanishes:*

$$
\mathcal{L}_{\delta_\xi}(E_{h_{aBB'}}^{\mathcal{A}_\mu}, E_{\psi_{aB}}^{\mathcal{A}'_\Lambda}) = 0, \quad \mathcal{L}_{\delta_\zeta}(E_{h_{aBB'}}^{\mathcal{A}_\mu}, E_{\psi_{aB}}^{\mathcal{A}'_\Lambda}) = 0, \quad \mathcal{L}_{\delta_\chi}(E_{h_{aBB'}}^{\mathcal{A}_\mu}, E_{\psi_{aB}}^{\mathcal{A}'_\Lambda}) = 0. \tag{2.18}
$$

These invariance equations assert geometrically that the gauge symmetries are tangent directions to the surface defined by solutions of the field equations in the space of spin-2 and spin-3/2 field configurations. Gauge invariance implies, consequently, that the commutations of the gauge symmetries have the same property.

Proposition 2. *Local gauge invariance holds only if the Lie derivative of the field equations with respect to the gauge symmetry commutators vanishes:*

$$\begin{aligned}
&\mathcal{L}_{[\delta_{\xi_1},\delta_{\xi_2}]}(E_{h_{aBB'}}^{\mathcal{A}_\mu}, E_{\psi_{aB}}^{\mathcal{A}'_\Lambda}) = 0, \quad \mathcal{L}_{[\delta_{\zeta_1},\delta_{\zeta_2}]}(E_{h_{aBB'}}^{\mathcal{A}_\mu}, E_{\psi_{aB}}^{\mathcal{A}'_\Lambda}) = 0, \quad \mathcal{L}_{[\delta_{\chi_1},\delta_{\chi_2}]}(E_{h_{aBB'}}^{\mathcal{A}_\mu}, E_{\psi_{aB}}^{\mathcal{A}'_\Lambda}) = 0,\\
&\mathcal{L}_{[\delta_{\xi_1},\delta_{\zeta_2}]}(E_{h_{aBB'}}^{\mathcal{A}_\mu}, E_{\psi_{aB}}^{\mathcal{A}'_\Lambda}) = 0, \quad \mathcal{L}_{[\delta_{\xi_1},\delta_{\chi_2}]}(E_{h_{aBB'}}^{\mathcal{A}_\mu}, E_{\psi_{aB}}^{\mathcal{A}'_\Lambda}) = 0, \quad \mathcal{L}_{[\delta_{\zeta_1},\delta_{\chi_2}]}(E_{h_{aBB'}}^{\mathcal{A}_\mu}, E_{\psi_{aB}}^{\mathcal{A}'_\Lambda}) = 0.
\end{aligned} \tag{2.19}$$

An expansion of these equations (2.18) and (2.19) in powers of the fields gives a hierarchy of determining equations whose solutions yield all allowed deformation terms in the field equations and gauge symmetries.

Compared with Ref. [18, 26], this framework for deformations of the linear abelian gauge theory of a set of ordinary spin-2 and spin-3/2 fields is more general in that it does not use the familiar choice of a symmetric tensor for the spin-2 field variables, corresponding to ${\sigma_{[b}}^{BB'}{h_{a]BB'}}^{\mathcal{A}_\mu} = 0$, as imposed by gauge fixing with the linearized local Lorentz symmetry. Indeed, the choice here of nonsymmetric spin-2 field variables ${h_{aBB'}}^{\mathcal{A}_\mu}$ allows the framework to encompass the related possibilities of deforming the local Lorentz symmetry on the fields ${h_{aBB'}}^{\mathcal{A}_\mu}, {\psi_{aB}}^{\mathcal{A}'_\Lambda}$, and of having nonlinear couplings that involve the skew part of the fields ${\sigma_{[b}}^{BB'}{h_{a]BB'}}^{\mathcal{A}_\mu}$.

Finally, it is important to remark that, as in Ref. [18, 26], no conditions are assumed or required on the possibilities allowed for the form of the commutators of the deformed gauge symmetries in the framework here. However, through the condition of gauge invariance of the Lagrangian, closure of the deformed gauge symmetries on the solution space of the deformed field equations will be seen to arise order by order, stemming from the fact that the abelian gauge symmetries generate all of the gauge freedom in the solutions of the linear field equations. Any deformation therefore automatically determines an associated infinitesimal gauge group structure.

B. Deformation results

The solutions of the determining equations (2.18) and (2.19) can be obtained by the methods used for the deformation analysis in Ref. [18, 19]. We now outline the steps for the corresponding analysis here. (See Ref. [27] for more details.) To begin, the first order parts of all allowed deformations are found by solving the 0th-order part of the Lie derivative

commutator equations (2.19) and 1st-order part of the Lie derivative equation (2.18) for, respectively, the linear terms in the gauge symmetries $\overset{(1)}{\delta} h_{aBB'}{}^{\mathcal{A}_\mu}$, $\overset{(1)}{\delta} \psi_{aB}{}^{\mathcal{A}'_\Lambda}$ and quadratic terms in the field equations $\overset{(2)}{E}{}_{h_{aBB'}}^{\mathcal{A}_\mu}, \overset{(2)}{E}{}_{\psi_{aB}}^{\mathcal{A}'_\Lambda}$. Calculation of the gauge symmetry commutators $\overset{(0)}{\delta}_1(\overset{(1)}{\delta}_2 h_{aBB'}{}^{\mathcal{A}_\mu}) - \overset{(0)}{\delta}_2(\overset{(1)}{\delta}_1 h_{aBB'}{}^{\mathcal{A}_\mu})$ and $\overset{(0)}{\delta}_1(\overset{(1)}{\delta}_2 \psi_{aB}{}^{\mathcal{A}'_\Lambda}) - \overset{(0)}{\delta}_2(\overset{(1)}{\delta}_1 \psi_{aB}{}^{\mathcal{A}'_\Lambda})$ then determines the lowest-order part of the infinitesimal gauge group structure $[\delta_1, \delta_2] \overset{(0)}{=} \overset{(0)}{\delta}_3$. Closure of this gauge group structure at the next lowest order is derived from equations given by the 1st-order part of the Lie derivative commutator equations (2.19) minus the 1st-order part of the Lie derivative equation (2.18) for the commutator gauge symmetries, where the field variables are taken to satisfy the linear field equations, $\overset{(1)}{E}{}_{h_{ab}}^{\mathcal{A}_\mu} = \overset{(1)}{E}{}_{\psi_{aB}}^{\mathcal{A}'_\Lambda} = 0$. When the gauge symmetry variables are taken to be rigid, $\xi_a{}^{\mathcal{A}_\mu} = \text{const.}$, $\zeta_B{}^{\mathcal{A}'_\Lambda} = \text{const.}$, $\chi_{AB}{}^{\mathcal{A}_\mu} = 0$, so that $\overset{(0)}{\delta} h_{aBB'}{}^{\mathcal{A}_\mu} = \overset{(0)}{\delta} \psi_{aB}{}^{\mathcal{A}'_\Lambda} = 0$, the resulting Lie derivative equations are seen to impose integrability conditions on the first-order parts of the deformations. In particular, algebraic conditions arise on the coupling constants in the linear terms in the gauge symmetries and quadratic terms in the field equations. These conditions are necessary (and, in fact, sufficient) to allow solving for the quadratic terms in the gauge symmetries $\overset{(2)}{\delta} h_{aBB'}{}^{\mathcal{A}_\mu}$, $\overset{(2)}{\delta} \psi_{aB}{}^{\mathcal{A}'_\Lambda}$ and cubic terms in the field equations $\overset{(3)}{E}{}_{h_{aBB'}}^{\mathcal{A}_\mu}, \overset{(3)}{E}{}_{\psi_{aB}}^{\mathcal{A}'_\Lambda}$ from, respectively, the 1st-order part of the Lie derivative commutator equations (2.19) and 2nd-order part of the Lie derivative equation (2.18) (with the gauge symmetry variables no longer being rigid and the field variables no longer satisfying the linear field equations), determining the second order parts of all allowed deformations. Last, uniqueness of the second and higher order parts of the deformations is considered by an induction argument. Let $\Delta \overset{(k)}{\delta} h_{aBB'}{}^{\mathcal{A}_\mu}$, $\Delta \overset{(k)}{\delta} \psi_{aB}{}^{\mathcal{A}'_\Lambda}$, $\Delta \overset{(k+1)}{E}{}_{h\ aBB'}^{\mathcal{A}_\mu}$, $\Delta \overset{(k+1)}{E}{}_{\psi\ aB}^{\mathcal{A}'_\Lambda}$ denote the difference of any two deformations that agree up to some finite order $1 \le k \le \ell$. These terms are shown to vanish at order $k = \ell + 1$ by solving the ℓth-order part of the Lie derivative commutator equation (2.19) and the $\ell + 1$st-order part of the Lie derivative equation (2.18). Hence it follows that any such deformations agree to all orders.

To state the main results of this analysis, it is convenient mathematically to view the set of ordinary spin-2 fields $h_{aBB'}{}^{\mathcal{A}_\mu}$ as being an equivalent single spin-2 field $\boldsymbol{h}_{aBB'}$ possessing the internal structure of a real vector space $\mathbb{R}^n \otimes \mathbf{X}$, and similarly to view the set of ordinary spin-3/2 fields $\psi_{aB}{}^{\mathcal{A}'_\Lambda}$ as being an equivalent single spin-3/2 field $\boldsymbol{\psi}_{aB}$ possessing the internal structure of a complexified vector space $\mathbb{R}^{n'} \otimes \mathbf{Y}$.

Theorem 1. *All non-higher-derivative deformations of the linear abelian gauge the-*

ory (2.2) to (2.7) for a set of ordinary spin-2 and spin-3/2 fields are (up to field redefinitions) equivalent to the nonlinear gauge theory of an algebra-valued tetrad field ${\boldsymbol{e}_a}^{BB'} = \mathbb{1}{\sigma_a}^{BB'} + {\boldsymbol{h}_a}^{BB'}$ *and an algebra-valued Rarita-Schwinger field* ${\boldsymbol{\varphi}_a}^{B} = {\boldsymbol{\psi}_a}^{B}$ *given by the chiral generalization of N= 1 supergravity theory based on a modified Grassmann algebra intertwined with charge conjugation. (Here* ${\sigma_a}^{BB'}$ *is a flat tetrad, and* $\mathbb{1}$ *is the unit element in the algebra.)*

The algebra on the vector spaces $\mathbb{R}^n \otimes \mathbf{X}, \mathbb{R}^{n'} \otimes \mathbf{Y}$, which we denote $\mathbb{A}_{\rm SG}$, underlying this nonlinear theory is defined by multiplication structure constants (real-valued) ${a^{\mathcal{A}_\mu}}_{\mathcal{B}_\alpha \mathcal{C}_\beta}$, and (complex-valued) ${b^{\mathcal{A}_\mu}}_{\mathcal{B}'_\Omega \mathcal{C}'_\Gamma}$, ${c^{\mathcal{A}'_\Lambda}}_{\mathcal{C}'_\Gamma \mathcal{B}_\alpha}$, ${d^{\mathcal{A}'_\Lambda}}_{\mathcal{B}_\alpha \mathcal{C}'_\Gamma}$. These constants satisfy the following linear and quadratic relations

$$
\begin{aligned}
{a^{\mathcal{A}_\mu}}_{\mathcal{B}_\alpha \mathcal{C}_\beta} &= {a^{\mathcal{A}_\mu}}_{\mathcal{C}_\beta \mathcal{B}_\alpha}, && (2.20)\\
{b^{\mathcal{A}_\mu}}_{\mathcal{B}'_\Omega \mathcal{C}'_\Gamma} &= -{\bar{b}^{\mathcal{A}_\mu}}_{\mathcal{C}'_\Gamma \mathcal{B}'_\Omega}, && (2.21)\\
{d^{\mathcal{A}'_\Lambda}}_{\mathcal{B}_\alpha \mathcal{C}'_\Gamma} &= {\bar{c}^{\mathcal{A}'_\Lambda}}_{\mathcal{C}_{\Gamma'} \mathcal{B}_\alpha}, && (2.22)
\end{aligned}
$$

and

$$
\begin{aligned}
{a^{\mathcal{A}_\mu}}_{\mathcal{B}_\alpha \mathcal{D}_\nu} {a^{\mathcal{D}_\nu}}_{\mathcal{C}_\beta \mathcal{E}_\sigma} &= {a^{\mathcal{A}_\mu}}_{\mathcal{C}_\beta \mathcal{D}_\nu} {a^{\mathcal{D}_\nu}}_{\mathcal{B}_\alpha \mathcal{E}_\sigma}, && (2.23)\\
{c^{\mathcal{A}'_\Lambda}}_{\mathcal{B}'_\Omega \mathcal{C}_\alpha} {c^{\mathcal{B}'_\Omega}}_{\mathcal{D}'_\Gamma \mathcal{E}_\mu} &= {c^{\mathcal{A}'_\Lambda}}_{\mathcal{D}'_\Gamma \mathcal{B}_\nu} {a^{\mathcal{B}_\nu}}_{\mathcal{C}_\alpha \mathcal{E}_\mu}, && (2.24)\\
{a^{\mathcal{A}_\mu}}_{\mathcal{D}_\nu \mathcal{B}_\alpha} {b^{\mathcal{D}_\nu}}_{\mathcal{C}'_\Lambda \mathcal{E}'_\Gamma} &= {b^{\mathcal{A}_\mu}}_{\mathcal{C}'_\Lambda \mathcal{B}'_\Omega} {c^{\mathcal{B}'_\Omega}}_{\mathcal{E}'_\Gamma \mathcal{B}_\alpha}, && (2.25)\\
{c^{\mathcal{A}'_\Lambda}}_{\mathcal{C}'_\Gamma \mathcal{D}_\nu} {b^{\mathcal{D}_\nu}}_{\mathcal{E}'_\Sigma \mathcal{B}'_\Omega} &= -{c^{\mathcal{A}'_\Lambda}}_{\mathcal{B}'_\Omega \mathcal{D}_\nu} {b^{\mathcal{D}_\nu}}_{\mathcal{E}'_\Sigma \mathcal{C}'_\Gamma}, && (2.26)
\end{aligned}
$$

together with the additional relations

$$
\begin{aligned}
& q_{\mathcal{A}_\mu \mathcal{B}_\nu} = q_{\mathcal{B}_\nu \mathcal{A}_\mu}, \quad q_{\mathcal{B}_\nu \mathcal{A}_\mu} {a^{\mathcal{A}_\mu}}_{\mathcal{C}_\alpha \mathcal{D}_\beta} = q_{\mathcal{C}_\alpha \mathcal{A}_\mu} {a^{\mathcal{A}_\mu}}_{\mathcal{B}_\nu \mathcal{D}_\beta}, && (2.27)\\
& {q'}_{\mathcal{A}'_\Lambda \mathcal{B}'_\Gamma} = -{\bar{q}'}_{\mathcal{B}'_\Gamma \mathcal{A}'_\Lambda}, \quad q_{\mathcal{B}_\nu \mathcal{A}_\mu} {b^{\mathcal{A}_\mu}}_{\mathcal{C}'_\Lambda \mathcal{D}'_\Gamma} = {q'}_{\mathcal{C}'_\Lambda \mathcal{E}'_\Omega} {c^{\mathcal{E}'_\Omega}}_{\mathcal{D}'_\Gamma \mathcal{B}_\nu}, && (2.28)
\end{aligned}
$$

where an overbar denotes complex conjugation. Note that, due to the equality (2.22), the structure constants ${d^{\mathcal{A}'_\Lambda}}_{\mathcal{B}_\alpha \mathcal{C}'_\Gamma}$ will be suppressed hereafter.

To conclude the main results, we note that Theorem 1 can be immediately specialized to the case of spin-2 fields alone.

Theorem 2. *All non-higher-derivative deformations of the linear abelian gauge theory (2.2), (2.4), (2.6), (2.7) for a set of ordinary spin-2 fields are (up to field redefinitions)*

equivalent to the nonlinear gauge theory given by the formulation of Einstein gravity theory for an algebra-valued metric field $\boldsymbol{g}_{ab} = \mathbb{1}\eta_{ab} + \sigma_{(b}{}^{BB'}\boldsymbol{h}_{a)BB'}$ using an even-Grassmann algebra.

In this nonlinear theory the even-Grassmann algebra, which we denote $\mathbb{A}_{\mathrm{G}}$ on the vector space $\mathbb{R}^n \otimes \mathbf{X}$, is defined by the multiplication structure constants (real-valued) $a^{\mathcal{A}_\mu}{}_{\mathcal{B}_\alpha\mathcal{C}_\beta}$ satisfying the linear and quadratic relations (2.20), (2.23), (2.27).

III. MAIN RESULTS

The results obtained in Theorem 1 for non-higher-derivative deformations are now recast to give a classification of nonlinear gauge theories of $n \geq 1$ spin-2 fields $h_{aBB'}{}^{\mu}$ ($\mu = 1, \dots, n$) and $n' \geq 1$ spin-3/2 fields $\psi_{aB}{}^{\Lambda}$ ($\Lambda = 1, \dots, n'$) with formal internal multiplication rules. This will be accomplished by factorizing the algebra structure on the vector spaces $\mathbb{R}^n \otimes \mathbf{X}$ and $\mathbb{R}^{n'} \otimes \mathbf{Y}$ defined by the coupling constants $a^{\mathcal{A}_\mu}{}_{\mathcal{B}_\alpha\mathcal{C}_\beta}$, $b^{\mathcal{A}_\mu}{}_{\mathcal{B}'_\Omega\mathcal{C}'_\Gamma}$, $c^{\mathcal{A}'_\Lambda}{}_{\mathcal{C}'_\Gamma\mathcal{B}_\alpha}$ associated with deformations for the equivalent set of ordinary fields $h_{aBB'}{}^{\mathcal{A}_\mu}, \psi_{aB}{}^{\mathcal{A}'_\Lambda}$.

A. Algebraic structure

To proceed, we consider the factorizations

$$a^{\mathcal{A}_\mu}{}_{\mathcal{B}_\alpha\mathcal{C}_\beta} = a^{\mu}{}_{\alpha\beta}A^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}}, \quad b^{\mathcal{A}_\mu}{}_{\mathcal{B}'_\Omega\mathcal{C}'_\Gamma} = b^{\mu}{}_{\Omega\Gamma}B^{\mathcal{A}}{}_{\mathcal{B}'\mathcal{C}'}, \quad c^{\mathcal{A}'_\Lambda}{}_{\mathcal{C}'_\Gamma\mathcal{B}_\alpha} = c^{\Lambda}{}_{\Gamma\alpha}C^{\mathcal{A}'}{}_{\mathcal{C}'\mathcal{B}}, \tag{3.1}$$

together with

$$q_{\mathcal{A}_\mu\mathcal{B}_\nu} = q_{\mu\nu}Q_{\mathcal{A}\mathcal{B}}, \quad q'_{\mathcal{A}'_\Lambda\mathcal{B}'_\Gamma} = q'_{\Lambda\Gamma}Q'_{\mathcal{A}'\mathcal{B}'}, \tag{3.2}$$

where $a^{\mu}{}_{\alpha\beta}$, $A^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}}$, $q_{\mu\nu}$, $Q_{\mathcal{A}\mathcal{B}}$ are real constants, and $b^{\mu}{}_{\Omega\Gamma}$, $B^{\mathcal{A}}{}_{\mathcal{B}'\mathcal{C}'}$, $c^{\Lambda}{}_{\Gamma\alpha}$, $C^{\mathcal{A}'}{}_{\mathcal{C}'\mathcal{B}}$, $q'_{\Lambda\Gamma}$, $Q'_{\mathcal{A}'\mathcal{B}'}$ are complex constants. Mathematically, this corresponds to factorizing the algebra $\mathbb{A}_{\mathrm{SG}}$ defined by (2.20) to (2.28) on $\mathbb{R}^n \otimes \mathbf{X}, \mathbb{R}^{n'} \otimes \mathbf{Y}$, into an internal part on $\mathbf{X} \oplus \mathbf{Y}$, denoted $\mathbb{A}_{\mathrm{internal}}$, with multiplication structure constants $A^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}}$, $B^{\mathcal{A}}{}_{\mathcal{B}'\mathcal{C}'}$, $C^{\mathcal{A}'}{}_{\mathcal{C}'\mathcal{B}}$, and a $n + n'$ dimensional part on $\mathbb{R}^n \oplus \mathbb{R}^{n'}$, denoted $\mathbb{A}_{\mathrm{coupling}}$, corresponding to the coupling constants $a^{\mu}{}_{\alpha\beta}$, $b^{\mu}{}_{\Omega\Gamma}$, $c^{\Lambda}{}_{\Gamma\alpha}$, for a set of n spin-2 fields and n' spin-3/2 fields. We now substitute the factorizations (3.1) and (3.2) into the linear and quadratic algebraic relations (2.20) to (2.28), which correspondingly factorize into the following parts:

$$a^{\mu}{}_{\alpha\beta} = \lambda_1 a^{\mu}{}_{\beta\alpha}, \quad A^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}} = 1/\lambda_1 A^{\mathcal{A}}{}_{\mathcal{C}\mathcal{B}}, \quad \lambda_1{}^2 = 1, \tag{3.3}$$

$$a^{\mu}{}_{\alpha\nu}a^{\nu}{}_{\beta\sigma} = \lambda_2 a^{\mu}{}_{\beta\nu}a^{\nu}{}_{\alpha\sigma}, \quad A^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}}A^{\mathcal{C}}{}_{\mathcal{D}\mathcal{E}} = 1/\lambda_2 A^{\mathcal{A}}{}_{\mathcal{D}\mathcal{C}}A^{\mathcal{C}}{}_{\mathcal{B}\mathcal{E}}, \quad {\lambda_2}^2 = 1, \tag{3.4}$$

and

$$b^{\mu}{}_{\Lambda\Gamma} = \lambda_3 \bar{b}^{\mu}{}_{\Gamma\Lambda}, \quad B^{\mathcal{A}}{}_{\mathcal{B}'\mathcal{C}'} = -1/\lambda_3 \bar{B}^{\mathcal{A}}{}_{\mathcal{C}'\mathcal{B}'}, \quad \lambda_3\bar{\lambda}_3 = 1, \tag{3.5}$$

$$c^{\Lambda}{}_{\Gamma\nu}b^{\nu}{}_{\Sigma\Omega} = \lambda_4 c^{\Lambda}{}_{\Omega\nu}b^{\nu}{}_{\Sigma\Gamma}, \quad C^{\mathcal{A}'}{}_{\mathcal{B}'\mathcal{C}}B^{\mathcal{C}}{}_{\mathcal{D}'\mathcal{E}'} = -1/\lambda_4 C^{\mathcal{A}'}{}_{\mathcal{E}'\mathcal{C}}B^{\mathcal{C}}{}_{\mathcal{D}'\mathcal{B}'}, \quad {\lambda_4}^2 = 1, \tag{3.6}$$

$$a^{\mu}{}_{\nu\alpha}b^{\nu}{}_{\Lambda\Gamma} = \lambda_5 b^{\mu}{}_{\Lambda\Omega}c^{\Omega}{}_{\Gamma\alpha}, \quad A^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}}B^{\mathcal{B}}{}_{\mathcal{D}'\mathcal{E}'} = 1/\lambda_5 B^{\mathcal{A}}{}_{\mathcal{D}'\mathcal{B}'}C^{\mathcal{B}'}{}_{\mathcal{E}'\mathcal{C}}, \quad \lambda_5 \neq 0, \tag{3.7}$$

$$c^{\Lambda}{}_{\Omega\alpha}c^{\Omega}{}_{\Gamma\beta} = \lambda_6 c^{\Lambda}{}_{\Gamma\nu}a^{\nu}{}_{\alpha\beta}, \quad C^{\mathcal{A}'}{}_{\mathcal{B}'\mathcal{C}}C^{\mathcal{B}'}{}_{\mathcal{D}'\mathcal{E}} = 1/\lambda_6 C^{\mathcal{A}'}{}_{\mathcal{D}'\mathcal{B}}A^{\mathcal{B}}{}_{\mathcal{C}\mathcal{E}}, \quad \lambda_6 \neq 0, \tag{3.8}$$

together with

$$q_{\mu\nu} = \lambda_7 q_{\nu\mu}, \quad Q_{\mathcal{A}\mathcal{B}} = 1/\lambda_7 Q_{\mathcal{B}\mathcal{A}}, \quad {\lambda_7}^2 = 1, \tag{3.9}$$

$$q_{\nu\mu}a^{\mu}{}_{\alpha\beta} = \lambda_8 q_{\alpha\mu}a^{\mu}{}_{\nu\beta}, \quad Q_{\mathcal{B}\mathcal{A}}A^{\mathcal{A}}{}_{\mathcal{C}\mathcal{D}} = 1/\lambda_8 Q_{\mathcal{C}\mathcal{A}}A^{\mathcal{A}}{}_{\mathcal{B}\mathcal{D}}, \quad {\lambda_8}^2 = 1, \tag{3.10}$$

and

$$q'_{\Lambda\Gamma} = \lambda_9 \bar{q}'_{\Gamma\Lambda}, \quad Q'_{\mathcal{A}'\mathcal{B}'} = -1/\lambda_9 \bar{Q}'_{\mathcal{B}'\mathcal{A}'}, \quad \lambda_9\bar{\lambda}_9 = 1, \tag{3.11}$$

$$q_{\nu\mu}b^{\mu}{}_{\Lambda\Gamma} = \lambda_{10} q_{\Lambda\Omega}c^{\Omega}{}_{\Gamma\nu}, \quad Q_{\mathcal{B}\mathcal{A}}B^{\mathcal{A}}{}_{\mathcal{C}'\mathcal{D}'} = 1/\lambda_{10} Q'_{\mathcal{C}'\mathcal{A}'}C^{\mathcal{A}'}{}_{\mathcal{D}'\mathcal{B}}, \quad \lambda_{10} \neq 0, \tag{3.12}$$

for some constants λ. Note that the relation (3.3) implies $a^{\mu}{}_{\alpha\beta}$ is either symmetric or antisymmetric in its lower indices. Compatibility of this index symmetry with the similar symmetry imposed by the relation (3.4) requires $\lambda_1 = \lambda_2$ (or else $a^{\mu}{}_{\alpha\nu}a^{\nu}{}_{\beta\sigma} = 0$ and $A^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}}A^{\mathcal{C}}{}_{\mathcal{D}\mathcal{E}} = 0$). The same argument applied to relations (3.9) and (3.10) leads to $\lambda_7 = \lambda_8$. Likewise, in the case where $b^{\mu}{}_{\Omega\Gamma}$ is taken to be real, then compatibility of the resulting index symmetries imposed by (3.5) and (3.6) requires $\lambda_3 = \lambda_4$ (or else $c^{\Lambda}{}_{\Gamma\nu}b^{\nu}{}_{\Sigma\Omega} = 0$ and $C^{\mathcal{A}'}{}_{\mathcal{B}'\mathcal{C}}B^{\mathcal{C}}{}_{\mathcal{D}'\mathcal{E}'} = 0$).

The main properties of the internal algebra on the vector spaces $\mathbf{X}, \mathbf{Y}$ and the field-coupling algebra on the vector spaces $\mathbb{R}^n, \mathbb{R}^{n'}$ defined by the algebraic equations (3.3) to (3.8) will now be summarized. In equations (3.3) and (3.4), the condition $\lambda_1 = \lambda_2 = \pm 1$ leads to two cases. For $\lambda_1 = \lambda_2 = 1$, $a^{\mu}{}_{\alpha\beta}$ defines multiplication structure constants of a real, commutative, associative algebra $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, while $A^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}}$ defines multiplication structure constants of a real, anticommutative, anti-associative algebra $\mathbf{X} \times \mathbf{X} \to \mathbf{X}$. These algebras exchange properties for $\lambda_1 = \lambda_2 = -1$. The multiplication laws consist of

$$u_1 u_2 = \pm u_2 u_1, \quad \varrho_1\varrho_2 = \mp \varrho_2\varrho_1, \tag{3.13}$$

$$u_1(u_2 u_3) = \pm(u_1 u_2)u_3, \quad \varrho_1(\varrho_2\varrho_3) = \mp(\varrho_1\varrho_2)\varrho_3, \tag{3.14}$$

where u's, ϱ's stand for elements in $\mathbb{R}^n, \mathbf{X}$.

In equations (3.5) and (3.6), consider first the situation of real algebras, arising if $b^{\mu}{}_{\Omega\Gamma} = \bar{b}^{\mu}{}_{\Omega\Gamma}$, $B^{\mathcal{A}}{}_{\mathcal{B}'\mathcal{C}'} = \bar{B}^{\mathcal{A}}{}_{\mathcal{B}'\mathcal{C}'}$ (which requires $\lambda_3 = \lambda_4$). Then, the condition $\lambda_3 = \lambda_4 = \pm 1$ leads to two cases. For $\lambda_3 = \lambda_4 = 1$, $b^{\mu}{}_{\Omega\Gamma}$ and $c^{\Lambda}{}_{\Gamma\alpha}$ define multiplication structure constants of real, commutative, associative products $\mathbb{R}^{n'} \times \mathbb{R}^{n'} \to \mathbb{R}^n$, $\mathbb{R}^{n'} \times \mathbb{R}^n \to \mathbb{R}^{n'}$, while $B^{\mathcal{A}}{}_{\mathcal{B}'\mathcal{C}'}$ and $C^{\mathcal{A}'}{}_{\mathcal{C}'\mathcal{B}}$ define multiplication structure constants of real, anticommutative, associative products $\mathbf{Y} \times \mathbf{Y} \to \mathbf{X}$ and $\mathbf{Y} \times \mathbf{X} \to \mathbf{Y}$. These products exchange properties for $\lambda_3 = \lambda_4 = -1$. The multiplication laws are given by

$$o_1 o_2 = \pm o_2 o_1, \quad \vartheta_1 \vartheta_2 = \mp \vartheta_2 \vartheta_1, \tag{3.15}$$

$$o_1(o_2 o_3) = \pm o_3(o_2 o_1) = o_3(o_1 o_2), \quad \vartheta_1(\vartheta_2 \vartheta_3) = \mp \vartheta_3(\vartheta_2 \vartheta_1) = \vartheta_3(\vartheta_1 \vartheta_2), \tag{3.16}$$

where o's, ϑ's stand for elements in $\mathbb{R}^{n'}, \mathbf{Y}$.

The situation of complex algebras arises from equations (3.5) and (3.6) if $b^{\mu}{}_{\Omega\Gamma} \neq \bar{b}^{\mu}{}_{\Omega\Gamma}$, $B^{\mathcal{A}}{}_{\mathcal{B}'\mathcal{C}'} \neq \bar{B}^{\mathcal{A}}{}_{\mathcal{B}'\mathcal{C}'}$. There is no relationship required between $\lambda_3 = e^{i\theta}$ and $\lambda_4 = \pm 1$ in this situation. These conditions then lead to two cases giving either associative or anti-associative multiplication rules (3.14) and also to a one-parameter family of cases of generalized commutative-type multiplication rules

$$o_1 o_2 = e^{\pm i\theta} \overline{\bar{o}_1 \bar{o}_2}, \quad \vartheta_1 \vartheta_2 = -e^{\mp i\theta} \overline{\bar{\vartheta}_1 \bar{\vartheta}_2}, \tag{3.17}$$

in terms of a real constant θ.

Finally, from equations (3.7) and (3.8), the previous products satisfy generalized associative-type multiplication rules

$$(o_1 o_2) u_1 = \lambda o_1(o_2 u_1), \quad (\vartheta_1 \vartheta_2) \varrho_1 = 1/\lambda \vartheta_1(\vartheta_2 \varrho_1), \tag{3.18}$$

$$(o_1 u_1) u_2 = \lambda' o_1(u_1 u_2), \quad (\vartheta_1 \varrho_1) \varrho_2 = 1/\lambda' \vartheta_1(\varrho_1 \varrho_2), \tag{3.19}$$

in terms of real constants $\lambda \neq 0, \lambda' \neq 0$. Thus, strict associativity holds for the case $\lambda = \lambda' = 1$, whereas anti-associativity holds for the case $\lambda = \lambda' = -1$. Other values of λ, λ' yield generalized types of associativity.

Hence the algebraic equations (3.3) to (3.8) together define an internal algebra $\mathbb{A}_{\text{internal}}$ on $\mathbf{X} \oplus \mathbf{Y}$ with formal multiplication rules (3.13) to (3.19) for the even and odd elements ϱ's and ϑ's, and a field-coupling algebra $\mathbb{A}_{\text{coupling}}$ on $\mathbb{R}^n \oplus \mathbb{R}^{n'}$ with multiplication structure (3.13) to (3.19) for the even and odd elements u's and o's.

The remaining algebraic equations (3.9) to (3.12) determine an inner product on the internal algebra and on the field-coupling algebra, as will be needed for construction of a scalar Lagrangian later. From equations (3.9) and (3.10), the independent conditions $\lambda_7 = \pm 1$, $\lambda_8 = \pm 1$ yield four cases for the symmetry and invariance properties of the inner products defined on $\mathbb{R}^n, \mathbf{X}$. For $\lambda_7 = \pm 1$, the inner products are respectively symmetric or antisymmetric

$$< u_1, u_2 >= \pm < u_2, u_1 >, \quad < \varrho_1, \varrho_2 >= \pm < \varrho_2, \varrho_1 >, \tag{3.20}$$

and for $\lambda_8 = \pm 1$, invariant or anti-invariant

$$< u_1, u_2 u_3 >= \pm < u_2, u_1 u_3 >, \quad < \varrho_1, \varrho_2 \varrho_3 >= \pm < \varrho_2, \varrho_1 \varrho_3 > . \tag{3.21}$$

Similarly, from equations (3.11) and (3.12), the inner products defined on $\mathbb{R}^{n'}, \mathbf{Y}$ satisfy a two-parameter family of cases of symmetry properties

$$< o_1, o_2 >= e^{\pm i\theta} \overline{< \bar{o}_2, \bar{o}_1 >}, \quad < \vartheta_1, \vartheta_2 >= -e^{\mp i\theta} \overline{< \bar{\vartheta}_2, \bar{\vartheta}_1 >}, \tag{3.22}$$

and invariance properties

$$< u_1, o_1 o_2 >= \lambda < o_1, o_2 u_1 >, \quad < \varrho_1, \vartheta_1 \vartheta_2 >= 1/\lambda < \vartheta_1, \vartheta_2 \varrho_1 >, \tag{3.23}$$

in terms of a real constant θ and complex constant $\lambda \neq 0$. In the situation of real inner products, which arise if ${q'}_{\Lambda\Gamma} = {\bar{q}'}_{\Lambda\Gamma}$, ${Q'}_{\mathcal{A}'\mathcal{B}'} = {\bar{Q}'}_{\mathcal{A}'\mathcal{B}'}$, the symmetry property (3.22) reduces to strict symmetry or antisymmetry

$$< o_1, o_2 >= \pm \overline{< \bar{o}_2, \bar{o}_1 >}, \quad < \vartheta_1, \vartheta_2 >= \mp \overline{< \bar{\vartheta}_2, \bar{\vartheta}_1 >} \tag{3.24}$$

while the invariance property (3.23) is restricted to $\lambda = \bar{\lambda}$.

B. Nonlinear gauge theories of spin-2 fields

To begin, consider for spin-2 fields alone the factorization of the even subalgebra of $\mathbb{A}_{\rm SG}$,

$$\mathbb{A}^{\rm spin2}_{\rm coupling} \otimes \mathbb{A}^{\rm spin2}_{\rm internal} \equiv \mathbb{A}_{\rm G} \tag{3.25}$$

defining a field-coupling algebra $\mathbb{A}^{\rm spin2}_{\rm coupling}$ for a set of $n \geq 1$ spin-2 fields ${h_{aBB'}}^{\mu}$ whose formal multiplication rules are given by an internal algebra $\mathbb{A}^{\rm spin2}_{\rm internal}$. This factorization combined with the deformation results in Theorem 2 yields a classification of nonlinear gauge theories of spin-2 fields ${h_{aBB'}}^{\mu}$ ($\mu = 1, \ldots, n$) with formal multiplication rules.

Theorem 3. *All non-higher-derivative gauge theories of a nonlinearly coupled set of $n \geq 1$ spin-2 fields ${h_{aBB'}}^{\mu}$ using formal multiplication rules are (up to field redefinitions) given by the formulation of Einstein gravity theory for a single matrix-algebra valued metric field ${g_{ab\mathcal{B}_\nu}}^{\mathcal{A}_\mu} = {\delta_{\mathcal{B}_\nu}}^{\mathcal{A}_\mu}\eta_{ab} + {a^{\mathcal{A}_\mu}}_{\mathcal{B}_\nu\mathcal{C}_\sigma}{\sigma_{(b}}^{BB'}{h_{a)BB'}}^{\mathcal{C}_\sigma}$ such that the algebra is a factorization of an associative, commutative algebra (3.25) as given by relations (3.3) and (3.4). The coupling constants and multiplication rules for ${h_{aBB'}}^{\mu}$ are given by the algebras $\mathbb{A}^{\rm spin2}_{\rm coupling}, \mathbb{A}^{\rm spin2}_{\rm internal}$ in this factorization.*

These nonlinear spin-2 gauge theories for ${h_{aBB'}}^{\mu}$ ($\mu = 1, \ldots, n$) are most naturally constructed in terms of metric tensor fields

$$ {g_{ab\nu}}^{\mu} = {a^{\mu}}_{\nu\sigma}{h_{ab}}^{\sigma} + {\delta_{\nu}}^{\mu}\eta_{ab} \tag{3.26} $$

and Christoffel connection tensors

$$ {{\Gamma^{c}}_{ab}}^{\mu} = {g^{-1cd\mu}}_{\nu}(\partial_{(a}{h_{b)d}}^{\nu} - \tfrac{1}{2}\partial_{d}{h_{ab}}^{\nu}) \tag{3.27} $$

where ${g^{-1ab\mu}}_{\nu}$ is the inverse of ${g_{ab\nu}}^{\mu}$ satisfying

$$ {g^{-1ac\mu}}_{\nu}{g_{ab\sigma}}^{\nu} = {\delta_{b}}^{c}{\delta_{\sigma}}^{\mu}. \tag{3.28} $$

Introduce the Riemann curvature associated with ${{\Gamma^{c}}_{ab}}^{\mu}$ by

$$ {R_{abc}}^{d\mu} = \partial_{[a}{{\Gamma^{d}}_{b]c}}^{\mu} + {a^{\mu}}_{\alpha\beta}{{\Gamma^{e}}_{c[a}}^{\alpha}{{\Gamma^{d}}_{b]e}}^{\beta}, \tag{3.29} $$

along with the Ricci tensor and scalar curvature

$$ {R_{ab}}^{\mu} = {R_{acb}}^{c\mu}, \quad R^{\mu} = {g^{-1ab\mu}}_{\nu}{R_{ab}}^{\nu}. \tag{3.30} $$

Then the field equations for ${h_{ab}}^{\mu}$ are given by the vacuum Einstein tensor

$$ {G_{ab}}^{\mu} = {R_{ab}}^{\mu} - \tfrac{1}{2}{g_{ab\nu}}^{\mu}R^{\nu} = 0. \tag{3.31} $$

The gauge invariance on solutions ${h_{ab}}^{\mu}$ is given by a general-covariance symmetry

$$ \delta_{\xi}{h_{ab}}^{\mu} = \xi^{c\sigma}\partial_{c}{g_{ab\sigma}}^{\mu} + 2\partial_{(a}\xi^{c\sigma}{g_{b)c\sigma}}^{\mu} \tag{3.32} $$

and also, trivially, by a local Lorentz symmetry

$$ \delta_{\chi}{h_{ab}}^{\mu} = 0 \tag{3.33} $$

since the skew part of ${h_{aBB'}}^{\mu}$ does not enter ${h_{ab}}^{\mu}$. A gauge invariant Lagrangian is readily formulated by considering the densitized scalar curvature

$$L^{\mu}_{\text{spin2}} = det^{\frac{1}{2}}(g)^{\mu}{}_{\nu} R^{\nu} \tag{3.34}$$

where $det^{\frac{1}{2}}(g)^{\mu}{}_{\nu}$ is the metric volume density given by

$$det^{\frac{1}{2}}(g)^{\mu}{}_{\nu} det^{\frac{1}{2}}(g)^{\nu}{}_{\sigma} = {g_{am\alpha}}^{\mu} {g_{bn\beta}}^{\alpha} {g_{ck\gamma}}^{\beta} {g_{dl\sigma}}^{\gamma} \epsilon^{abcd} \epsilon^{mnkl}. \tag{3.35}$$

Under the general-covariance symmetry (3.32), L^{μ}_{spin2} is invariant (to within a total divergence $\partial_a S^{a\mu}$) and in addition is trivially invariant with respect to the local Lorentz symmetry (3.33). Moreover, by variation of ${h_{ab}}^{\mu}$, L^{μ}_{spin2} yields the field equations (3.31), such that $\delta_h L^{\mu}_{\text{spin2}} = a^{\mu}{}_{\nu\sigma} G^{ab\nu} \delta {h_{ab}}^{\sigma} +$ a total divergence. Construction of an equivalent scalar Lagrangian will be addressed shortly. It is important to emphasize here that, in the expressions for the Lagrangian, field equations, and gauge symmetries, all products of the fields ${h_{ab}}^{\mu}$ involve formal multiplication given by the rules of the internal algebra $\mathbb{A}^{\text{spin2}}_{\text{internal}}$ (so any change in the order or arrangement of the fields requires using these formal rules). As a consequence, mathematically, these expressions are not real-valued but rather are regarded formally as taking values in $\mathbb{A}^{\text{spin2}}_{\text{internal}}$.

From the algebraic results in Sec. III A, we see that there are only two types allowed for the multiplication rules $\mathbb{A}^{\text{spin2}}_{\text{internal}}$ and the accompanying coupling constants $\mathbb{A}^{\text{spin2}}_{\text{coupling}}$ in the nonlinear theories (3.26) to (3.34). One type is, obviously, where $\mathbb{A}^{\text{spin2}}_{\text{coupling}}$ is an arbitrary n dimensional associative, commutative algebra and $\mathbb{A}^{\text{spin2}}_{\text{internal}}$ consists of standard multiplication so each ${h_{ab}}^{\mu}$ is formally a commuting field. In this situation, a simple case for $\mathbb{A}^{\text{spin2}}_{\text{internal}}$ is just given by the algebra of real numbers, $\mathbb{R}$, and correspondingly, ${h_{ab}}^{\mu}$ is a set of ordinary (i.e. real-valued) fields. The nonlinear theories (3.26) to (3.34) thereby are precisely the same as the multi-graviton theories first found by Cutler and Wald. This is most easily seen by employing a unit element $\mathbb{1}^{\mu}$ in $\mathbb{A}^{\text{spin2}}_{\text{coupling}}$ (appending one if none exists [16]). Then, through the relations $a^{\mu}{}_{\nu\sigma} \mathbb{1}^{\sigma} = {\delta_{\nu}}^{\mu}$ and ${g_{ab\nu}}^{\mu} = a^{\mu}{}_{\nu\sigma} {g_{ab}}^{\sigma}$, the nonlinear theories (3.26) to (3.34) simplify to Einstein gravity theory for the $\mathbb{A}^{\text{spin2}}_{\text{coupling}}$-valued metric tensor

$$\boldsymbol{g}_{ab} = \mathbb{1}\eta_{ab} + \sigma_{(b}{}^{BB'} \boldsymbol{h}_{a)BB'} \tag{3.36}$$

with $\mathbb{A}^{\text{spin2}}_{\text{internal}} = \mathbb{R}$.

The other allowed type is instead that $\mathbb{A}^{\text{spin2}}_{\text{coupling}}$ is an arbitrary n dimensional anticommutative, anti-associative algebra and $\mathbb{A}^{\text{spin2}}_{\text{internal}}$ comprises formal rules of anticommutative,

anti-associative multiplication. In this situation the fields ${h_{ab}}^{\mu}$ are each formally anticommuting and obey an anti-associativity relation for formal products with three (or more) fields,

$$\varrho_1\varrho_2 = -\varrho_2\varrho_1, \quad \varrho_1(\varrho_2\varrho_3) = -(\varrho_1\varrho_2)\varrho_3, \tag{3.37}$$

where ϱ's stand for ${h_{ab}}^{\mu}$ or products of any number of ${h_{ab}}^{\mu}$'s. An anticommutative, anti-associative field-coupling algebra $\mathbb{A}^{\rm spin2}_{\rm coupling}$ is characterized by structure constants with the antisymmetry properties

$$a^{\mu}{}_{(\alpha\beta)} = 0, \quad a^{\mu}{}_{\nu(\alpha}a^{\nu}{}_{\beta)\sigma} = 0. \tag{3.38}$$

A simple nontrivial example is given by $a^4{}_{1\ 2} \neq 0$, $a^5{}_{1\ 3} \neq 0$, $a^6{}_{2\ 3} \neq 0$, $a^7{}_{3\ 4} = a^7{}_{1\ 6} = -a^7{}_{2\ 5} \neq 0$ (all others zero, taking into account antisymmetry) with $n = 7$, producing a seven-dimensional anticommutative, anti-associative algebra $\mathbb{A}^{\rm spin2}_{\rm coupling}$. (This example arises by forming all possible anticommutative, anti-associative products of three generators: $u_1, u_2, u_3, u_{12} \equiv u_1u_2 = -u_2u_1, u_{13} \equiv u_1u_3 = -u_3u_1, u_{23} \equiv u_2u_3 = -u_3u_2, u_{123} \equiv u_1u_{23} = -u_{23}u_1 = -u_2u_{13} = u_{13}u_2 = u_3u_{12} = -u_{12}u_3$, and all other products equal to zero.) These antisymmetric structure constants $a^{\mu}{}_{\alpha\beta}$ yield a nonlinear multi-graviton theory (3.26) to (3.34) for a set of seven coupled anticommuting spin-2 fields ${h_{aBB'}}^{\mu}$ ($\mu = 1, \ldots, 7$). As illustrated by this example, the anticommuting nature (and anti-associativity) of ${h_{aBB'}}^{\mu}$ here is tied to the number n of spin-2 fields being at least two.

Moreover, in contrast to the commutative, associative type of algebra for $\mathbb{A}^{\rm spin2}_{\rm coupling}$ in the Cutler-Wald multi-graviton theories, the new multi-graviton theories presented here for an anticommutative, anti-associative type of algebra $\mathbb{A}^{\rm spin2}_{\rm coupling}$ do not have a formulation in terms of an $\mathbb{A}^{\rm spin2}_{\rm coupling}$-valued metric tensor (3.36), since no anticommutative algebra can admit a unit element.

Finally, we turn to constructing a scalar Lagrangian for these multi-graviton theories. The construction makes use of an invariant inner product on the field-coupling algebra $\mathbb{A}^{\rm spin2}_{\rm coupling}$ as follows.

Consider, first of all, the case of an associative, commutative algebra $\mathbb{A}^{\rm spin2}_{\rm coupling}$ on $\mathbb{R}^n$, with structure constants $a^{\mu}{}_{\alpha\beta}$. For simplicity, it is sufficient to assume $a^{\mu}{}_{\alpha\beta}$ is irreducible (namely, that the algebra is not a direct sum of two or more nontrivial subalgebras). An invariant symmetric inner product with nondegenerate components $q_{\mu\nu} = q_{(\mu\nu)}$ is characterized by

the invariance relation

$$q_{\mu[\nu} a^{\mu}{}_{\alpha]\beta} = 0. \tag{3.39}$$

Now, suppose $a^{\mu}{}_{\alpha\beta}$ possesses a unit element $\mathbb{1}^{\mu}$. Then the relation (3.39) shows that

$$q_{\nu\sigma} = q^{*}{}_{\mu} a^{\mu}{}_{\nu\sigma} \tag{3.40}$$

where $q^{*}{}_{\mu} = q_{\mu\nu}\mathbb{1}^{\nu}$ represents the dual of $\mathbb{1}^{\nu}$ with respect to the inner product. Consequently, any inner product is obtained by fixing some constants $q^{*}{}_{\mu}$ such that the components (3.40) are nondegenerate. If there is a nilpotent element in the algebra, then nondegeneracy holds if and only if the most nilpotent ideal is one-dimensional and $q^{*}{}_{\mu}$ is chosen to be nonvanishing on a most nilpotent element. An example of such algebras is any even-Grassmann algebra, $\mathbb{G}^{\text{even}}_{\text{unit}}$. Alternatively, if there are no nilpotent elements in the algebra, then due to the irreducibility assumption [16], the algebra is necessarily isomorphic to $\mathbb{R}$ or $\mathbb{C}$ and then it is sufficient to choose $q^{*}{}_{\mu}$ to be nonvanishing on the unit element.

Proposition 3. *For an associative, commutative algebra $\mathbb{A}^{\text{spin2}}_{\text{coupling}}$ with a unit element and with a most nilpotent ideal of at most one dimension, a scalar Lagrangian for the nonlinear theories (3.26) to (3.34) using formal associative, commutative multiplication rules for $h_{ab}{}^{\mu}$ is obtained simply by taking*

$$L_{\text{spin2}} = q^{*}{}_{\mu} L^{\mu}_{\text{spin2}} \tag{3.41}$$

where $q^{*}{}_{\mu}$ projects onto the most nilpotent element or, if none, the unit element.

This Lagrangian is invariant (to within a total divergence) under the gauge symmetries (3.32) and (3.33) and yields the field equations (3.31) by variation of $h_{ab}{}^{\mu}$.

Next suppose $\mathbb{A}^{\text{spin2}}_{\text{coupling}}$ is an associative, commutative algebra that is completely nilpotent. In this case, for any constants $q^{*}{}_{\mu}$, clearly the components (3.40) are degenerate on the most nilpotent ideal and so the scalar Lagrangian (3.41) breaks down as it does not contain (at least) the field $q^{*}{}_{\mu} h_{ab}{}^{\mu}$. Of course, the relation (3.40) is merely sufficient, but not necessary, for $q_{\nu\sigma}$ to satisfy the invariance property (3.39) when $\mathbb{A}^{\text{spin2}}_{\text{coupling}}$ does not possess a unit element. However, for the example of nilpotent even-Grassmann algebras, $\mathbb{G}^{\text{even}}_{\text{nilpotent}}$, any invariant $q_{\nu\sigma}$ is degenerate on the most nilpotent ideal. (For instance, consider the even-Grassmann algebra with three nilpotent generators: $u_1, u_2, u_3, u_{12} \equiv u_1u_2, u_{13} \equiv u_1u_3, u_{23} \equiv u_2u_3, u_{123} \equiv u_1u_2u_3$. Since the square of any element is zero, the invariance property of $q_{\nu\sigma}$ applied to $< u_{123}, u >=< u_1, uu_{23} >=< u_2, uu_{13} >=< u_3, uu_{12} >$ forces this to vanish

for all elements u.) An example on the other hand where an invariant nondegenerate $q_{\nu\sigma}$ exists is any nilpotent monogenic associative, commutative algebra, $\mathbb{V}_{\rm p}$. (Namely, consider an associative, commutative algebra that is generated by powers of a single element u such that $u^{p+1} = 0$. Then the inner product defined by $< u^j, u^k >= \delta_p{}^{j+k}$ is, clearly, invariant and nondegenerate on the entire algebra.) In this case, a scalar Lagrangian is constructed by the following procedure.

Proposition 4. *For a completely nilpotent associative, commutative algebra $\mathbb{A}^{\rm spin2}_{\rm coupling}$ with an invariant nondegenerate inner product, a scalar Lagrangian for the nonlinear theories (3.26) to (3.34) using formal associative, commutative multiplication rules for $h_{ab}{}^{\mu}$ is given in terms of components $q_{\mu\nu}$ of the inner product by*

$$L_{\rm spin2} = q_{\mu\nu} L^{\mu\nu} \tag{3.42}$$

where $L^{\mu\nu}$ is the coefficient of $a^{\sigma}{}_{\mu\nu}$ in $L^{\sigma}_{\rm spin2}$.

This construction works because, if we split $L^{\sigma}_{\rm spin2} = a^{\sigma}{}_{\mu\nu} L^{\mu\nu} +$ linear $h_{ab}{}^{\sigma}$ terms, the linear terms are found to be a total divergence and so $q_{\mu\nu} L^{\mu\nu}$ retains the gauge symmetries and field equations of $L^{\sigma}_{\rm spin2}$ due to the invariance property of $q_{\mu\nu}$.

Last, consider the case of an anticommutative, anti-associative algebra $\mathbb{A}^{\rm spin2}_{\rm coupling}$ on $\mathbb{R}^n$, with structure constants $a^{\mu}{}_{\alpha\beta}$. An invariant antisymmetric inner product with nondegenerate components $q_{\mu\nu} = q_{[\mu\nu]}$ is characterized by the invariance relation

$$q_{\mu(\nu} a^{\mu}{}_{\alpha)\beta} = 0. \tag{3.43}$$

Observe, here, that due to anticommutativity, the square of any element in $\mathbb{A}^{\rm spin2}_{\rm coupling}$ vanishes and hence the algebra is completely nilpotent. Consequently, although the relation (3.40) continues to yield an invariant inner product, it is degenerate on all most nilpotent elements. Moreover, an argument similar to the one for the example of a nilpotent even-Grassmann algebra $\mathbb{G}^{\rm even}_{\rm nilpotent}$ considered previously shows that any invariant inner product for an anticommutative, anti-associative algebra generated analogously to $\mathbb{G}^{\rm even}_{\rm nilpotent}$ must always be degenerate. However, if an extra element is appended in a suitable manner, then an invariant nondegenerate inner product exists on the enlarged algebra. (In particular, for the example of the seven-dimensional anticommutative, anti-associative algebra with three generators u_1, u_2, u_3 given earlier, the nilpotency property $u_1{}^2 = u_2{}^2 = u_3{}^2 = 0$ combined with the

invariance property $< u_{123}, u >=< u_1, uu_{23} >=< u_2, u_{13}u >=< u_3, uu_{12} >$ forces this inner product to vanish for all elements u. But, by appending an extra element u_0 and imposing $u_0u_{123} = 0$, it follows that $< u_{123}, u_0 >=< u_1, u_0u_{23} >=< u_2, u_{13}u_0 >=< u_3, u_0u_{12} >\neq 0$ determines an invariant nondegenerate inner product.) More generally, if we mod out with respect to the most nilpotent ideal in an anticommutative, anti-associative algebra formed from any number of generators, then the quotient algebra admits an invariant nondegenerate inner product.

The construction of a scalar Lagrangian in this case now parallels Proposition 4.

Proposition 5. *For an anticommutative, anti-associative algebra* $\mathbb{A}^{\rm spin2}_{\rm coupling}$ *with an invariant nondegenerate inner product, a scalar Lagrangian for the nonlinear theories (3.26) to (3.34) using formal anticommutative, anti-associative multiplication rules for* ${h_{ab}}^\mu$ *is given by (3.42) in terms of components* $q_{\mu\nu}$ *of the inner product.*

In all cases, the Lagrangians (3.34), (3.41), (3.42) are polynomial in ${h_{ab}}^\mu$ (and its derivatives) when, and only when, the field-coupling algebra $\mathbb{A}^{\rm spin2}_{\rm coupling}$ is completely nilpotent. Moreover, in this situation these Lagrangians depend essentially on the background flat tetrad $\sigma^{aBB'}$. Indeed, the nonlinear theories (3.26) to (3.34) are not independent of this flat tetrad unless the field-coupling algebra possesses a unit element.

As a final remark it is worth noting that the inner products on $\mathbb{A}^{\rm spin2}_{\rm coupling}$ underlying these scalar Lagrangians are of indefinite sign [22, 26, 27] whenever the field-coupling algebra is nontrivial (in particular, if the most nilpotent ideal is a proper subalgebra). Consequently, the canonical stress-energy tensor derived from the scalar Lagrangian associated to the set of coupled fields ${h_{ab}}^\mu$ lacks any formal positivity properties, in contrast to the well-known dominant energy property of the stress-energy tensor in the case of a single field h_{ab}.

C. Supersymmetric extensions

It is straightforward to construct supersymmetric extensions of the nonlinear multi-graviton theories (3.26) to (3.34), starting from a factorization of the algebra $\mathbb{A}_{\rm SG} = \mathbb{A}_{\rm coupling} \otimes \mathbb{A}_{\rm internal}$ to obtain a field-coupling algebra $\mathbb{A}_{\rm coupling}$ for a set of $n \geq 1$ spin-2 fields ${h_{aBB'}}^\mu$ ($\mu = 1, \ldots, n$) and $n' \geq 1$ spin-3/2 fields ${\psi_{aB}}^\Lambda$ ($\Lambda = 1, \ldots, n'$) whose formal multiplication rules are given by an internal algebra $\mathbb{A}_{\rm internal}$.

From the algebraic results in Sec. III A, we see that the allowed multiplication rules $\mathbb{A}_{\text{internal}}$ and accompanying coupling constants $\mathbb{A}_{\text{coupling}}$ comprise a rich variety of types. First of all, note the algebras possess a semidirect product structure stemming from a natural grading into even and odd parts, $\mathbb{A}^{\text{even/odd}}_{\text{coupling}}$ and $\mathbb{A}^{\text{even/odd}}_{\text{internal}}$, which correspond respectively to the spin-2 fields (assigned even-grading) and the spin-3/2 fields (assigned odd-grading). The even parts $\mathbb{A}^{\text{even}}_{\text{coupling}}$ and $\mathbb{A}^{\text{even}}_{\text{internal}}$ are given by the two types of algebras $\mathbb{A}_{\text{G}}$ discussed in Sec. III B for the case of spin-2 fields alone. The odd parts $\mathbb{A}^{\text{odd}}_{\text{coupling}}$ and $\mathbb{A}^{\text{odd}}_{\text{internal}}$ involve the product structure $\mathbb{A}^{\text{odd}}_{\text{SG}} \times \mathbb{A}^{\text{odd}}_{\text{SG}}$ into $\mathbb{A}^{\text{even}}_{\text{SG}}$, called odd multiplication, and $\mathbb{A}^{\text{odd}}_{\text{SG}} \times \mathbb{A}^{\text{even}}_{\text{SG}}$ into $\mathbb{A}^{\text{odd}}_{\text{SG}}$, called even-odd multiplication, which defines the semidirect product of $\mathbb{A}^{\text{even}}_{\text{SG}}$ with $\mathbb{A}^{\text{odd}}_{\text{SG}}$. In addition, $\mathbb{A}^{\text{odd}}_{\text{SG}}$ is allowed to be complexified whereas $\mathbb{A}^{\text{even}}_{\text{SG}}$ is necessarily real, with complex conjugation representing charge conjugation on the spin-2 and spin-3/2 fields.

In the case of real algebras $\mathbb{A}_{\text{SG}}$, the only allowed types of odd multiplication are associative and either anticommutative or commutative, while the allowed even-odd multiplication exhibits a two-parameter type of generalized associativity. The case of complexified algebras $\mathbb{A}_{\text{SG}}$ is richer in allowed types, due to intertwining of complex conjugation with both odd multiplication and even-odd multiplication. This will not be explored in further detail here.

To proceed, the supersymmetric extension of the nonlinear theories (3.26) to (3.34) for a set of fields ${h_{aBB'}}^{\mu}$ ($\mu = 1, \ldots, n$) and ${\psi_{aB}}^{\Lambda}$ ($\Lambda = 1, \ldots, n'$) with coupling constants given by $\mathbb{A}_{\text{coupling}}$ and using formal internal multiplication rules of $\mathbb{A}_{\text{internal}}$ is based on the following main result.

Theorem 4. *All non-higher-derivative gauge theories of a nonlinearly coupled set of $n \geq 1$ spin-2 fields ${h_{aBB'}}^{\mu}$ and $n' \geq 1$ spin-3/2 fields ${\psi_{aB}}^{\Lambda}$ using formal multiplication rules are (up to field redefinitions) given by the chiral generalization of N= 1 supergravity theory [21] for a single real, matrix-algebra valued tetrad field ${e_{aBB'\mathcal{B}_\nu}}^{\mathcal{A}_\mu} = {\delta_{\mathcal{B}_\nu}}^{\mathcal{A}_\mu}\sigma_{aBB'} + {a^{\mathcal{A}_\mu}}_{\mathcal{B}_\nu\mathcal{C}_\sigma}{h_{aBB'}}^{\mathcal{C}_\sigma}$ and a single conjugate-pair of complex, matrix-algebra valued vector-spinor fields ${\varphi_{aB\mathcal{B}_\nu}}^{\mathcal{A}'_\Lambda} = {c^{\mathcal{A}'_\Lambda}}_{\mathcal{C}'_\Gamma\mathcal{B}_\nu}{\psi_{aB}}^{\mathcal{C}'_\Gamma}$, ${\bar{\varphi}_{aB'\mathcal{C}'_\Gamma}}^{\mathcal{A}_\mu} = {b^{\mathcal{A}_\mu}}_{\mathcal{B}'_\Lambda\mathcal{C}'_\Gamma}{\bar{\psi}_{aB'}}^{\mathcal{B}'_\Lambda}$ such that the algebra is a factorization of a modified associative, graded-commutative algebra $\mathbb{A}_{\text{SG}} = \mathbb{A}_{\text{coupling}} \otimes \mathbb{A}_{\text{internal}}$ that intertwines with charge conjugation as given by relations (3.3) to (3.8). The coupling constants and multiplication rules for ${h_{aBB'}}^{\mu}$ and ${\psi_{aB}}^{\Lambda}$ are given by the algebras $\mathbb{A}_{\text{coupling}}, \mathbb{A}_{\text{internal}}$ in this factorization.*

The resulting nonlinear gauge theories for the fields ${h_{aBB'}}^{\mu}, {\psi_{aB}}^{\Lambda}$ ($\mu = 1, \ldots, n$, $\Lambda =$

$1, \ldots, n')$ are most naturally formulated in terms of vector-spinors

$$\varphi_{aB\nu}{}^{\Lambda} = c^{\Lambda}{}_{\Gamma\nu}\psi_{aB}{}^{\Gamma}, \quad \bar{\varphi}_{aB'\Gamma}{}^{\mu} = b^{\mu}{}_{\Lambda\Gamma}\bar{\psi}_{aB'}{}^{\Lambda}, \tag{3.44}$$

and tetrads

$$e_{aBB'\nu}{}^{\mu} = a^{\mu}{}_{\nu\sigma}h_{aBB'}{}^{\sigma} + \delta_{\nu}{}^{\mu}\sigma_{aBB'}, \quad e_{aBB'\Gamma}{}^{\Lambda} = c^{\Lambda}{}_{\Gamma\sigma}h_{aBB'}{}^{\sigma} + \delta_{\Gamma}{}^{\Lambda}\sigma_{aBB'}, \tag{3.45}$$

along with Lorentz spin connections and curvatures

$$\omega_{a}{}^{CD\mu} = \omega_{a}{}^{(CD)\mu}, \quad R_{ab}{}^{CD\mu} = \partial_{[a}\omega_{b]}{}^{CD\mu} + a^{\mu}{}_{\alpha\beta}\omega_{[a}{}^{CE\alpha}\omega_{b]E}{}^{D\beta} \tag{3.46}$$

determined by the torsion equation

$$\partial_{[b}e^{DD'}{}_{c]\nu}{}^{\mu} - a^{\mu}{}_{\nu\sigma}\omega_{[b|B}{}^{D\alpha}e^{BD'}{}_{|c]\alpha}{}^{\sigma} + c.c. = -\bar{\varphi}^{D'}_{[b|\Gamma}{}^{\mu}\varphi^{D}_{|c]\nu}{}^{\Gamma}. \tag{3.47}$$

This equation can be solved for $\omega_{a}{}^{CD\mu}$ by using the inverse tetrads which satisfy

$$e_{aCC'\sigma}{}^{\nu}e^{-1aBB'\mu}{}_{\nu} = \epsilon_{B}{}^{C}\epsilon_{B'}{}^{C'}\delta_{\sigma}{}^{\mu}, \quad e_{aCC'\Omega}{}^{\Gamma}e^{-1aBB'\Lambda}{}_{\Gamma} = \epsilon_{B}{}^{C}\epsilon_{B'}{}^{C'}\delta_{\Omega}{}^{\Lambda}, \tag{3.48}$$

where ϵ_{BC} is the spin metric. The field equations for $h_{aBB'}{}^{\mu}$ and $\psi_{aB}{}^{\Lambda}$ are given by the Einstein equation with spin-3/2 matter source

$$G^{aBB'\mu} - T^{aBB'\mu} = 0, \tag{3.49}$$

and by the Rarita-Schwinger equation

$$F_{[bc|D}{}^{\Gamma}e^{DD'}{}_{|a]\Gamma}{}^{\Lambda} = 0, \tag{3.50}$$

where

$$F_{cdB}{}^{\Lambda} = \partial_{[c}\psi_{d]B}{}^{\Lambda} + \varphi_{[d|E\sigma}{}^{\Lambda}\omega_{|c]B}{}^{E\sigma} \tag{3.51}$$

is the covariant Rarita-Schwinger field strength, and where

$$G^{aBB'\mu} = e^{BB'}{}_{b\nu}{}^{\mu}(R^{ab\nu} - \tfrac{1}{2}g^{-1ab\nu}{}_{\sigma}R^{\sigma}) \tag{3.52}$$

is the spinorial Einstein tensor for the metric fields

$$g_{ab\nu}{}^{\mu} = e_{aBB'\sigma}{}^{\mu}e^{BB'}{}_{b\nu}{}^{\sigma}, \tag{3.53}$$

with Ricci tensor $R^{ab\nu}$ and scalar curvature R^{σ} given by (3.30), and

$$T^{aBB'\mu} = \tfrac{1}{2}ie^{abcd\mu}{}_{\nu}\bar{\varphi}^{B'\nu}_{b\Gamma}F^{B\,\Gamma}_{cd} + c.c. \tag{3.54}$$

is the spinorial stress-energy tensor. Here

$$e_{abcd\nu}{}^{\mu} = det(e)^{\mu}{}_{\nu}\epsilon_{\ abcd} = 2ie^{A}{}_{[a|A'\alpha|}{}^{\mu}e^{A'}{}_{b|B\beta|}{}^{\alpha}e^{B}{}_{c|C'\gamma|}{}^{\beta}e^{C'}{}_{d]A\nu}{}^{\gamma} \tag{3.55}$$

is the tetrad volume tensor, with

$$det(e)^{\mu}{}_{\nu} = det^{\frac{1}{2}}(g)^{\mu}{}_{\nu}. \tag{3.56}$$

Gauge invariance on solutions $h_{aBB'}{}^{\mu}$ and $\psi_{aB}{}^{\Lambda}$ is given by a general-covariance symmetry

$$\delta_{\xi}h_{aBB'}{}^{\mu} = \xi^{c\sigma}\partial_{c}e_{aBB'\sigma}{}^{\mu} + \partial_{a}\xi^{c\sigma}e_{cBB'\sigma}{}^{\mu}, \quad \delta_{\xi}\psi_{aB}{}^{\Lambda} = \xi^{c\sigma}\partial_{c}\varphi_{aB\sigma}{}^{\Lambda} + \partial_{a}\xi^{c\sigma}\varphi_{cB\sigma}{}^{\Lambda}, \tag{3.57}$$

as well as a local Lorentz symmetry

$$\delta_{\chi}h_{aBB'}{}^{\mu} = \chi_{B}{}^{C\nu}e_{aCB'\nu}{}^{\mu} + c.c., \quad \delta_{\chi}\psi_{aB}{}^{\Lambda} = \varphi_{aC\sigma}{}^{\Lambda}\chi_{B}{}^{C\sigma}, \tag{3.58}$$

and a supersymmetry

$$\delta_{\zeta}h_{aBB'}{}^{\mu} = \bar{\varphi}_{aB'\Gamma}{}^{\mu}\zeta_{B}{}^{\Gamma} + c.c., \quad \delta_{\zeta}\psi_{aB}{}^{\Lambda} = \partial_{a}\zeta_{B}{}^{\Lambda} + c^{\Lambda}{}_{\Gamma\sigma}\zeta_{c}{}^{\Gamma}\omega_{aB}{}^{C\sigma}, \tag{3.59}$$

for arbitrary vector fields $\xi^{c\sigma}$ and arbitrary spinor fields $\zeta_{B}{}^{\Gamma}$ and $\chi_{BC}{}^{\nu} = \chi_{(BC)}{}^{\nu}$. A gauge invariant Lagrangian is obtained by considering the densitized sum of a gravitational scalar curvature term and matter current term

$$L^{\mu}_{\text{spin2,3/2}} = \epsilon^{\ abcd}\tfrac{1}{4}i\Big(2(e_{cCC'\nu}{}^{\mu}e^{C'}{}_{Dd\sigma}{}^{\nu})R_{ab}{}^{CD\sigma} + \bar{\varphi}_{aB\Gamma}{}^{\mu}(F_{cdB}{}^{\Omega}e^{BB'}{}_{b\Omega}{}^{\Gamma})\Big) + c.c.\ . \tag{3.60}$$

Under the symmetries (3.57) to (3.59), $L^{\mu}_{\text{spin2,3/2}}$ is invariant (to within a total divergence $\partial_{a}S^{a\mu}$). In addition, by variation of $h_{aBB'}{}^{\mu}$ and $\psi_{aB}{}^{\Lambda}$, $L^{\mu}_{\text{spin2,3/2}}$ yields the field equations (3.49) and (3.50). Note that all products of the fields $h_{aBB'}{}^{\mu}$ and $\psi_{aB}{}^{\Lambda}$ in the Lagrangian, field equations, and gauge symmetries involve formal multiplication rules given by $\mathbb{A}_{\text{internal}}$ (which must be used for reordering or rearranging of products). Consequently, these expressions are to be regarded formally as taking values in $\mathbb{A}_{\text{internal}}$.

When $\mathbb{A}_{\text{coupling}}$ is an arbitrary $n + n'$ dimensional commutative, associative graded algebra and $\mathbb{A}_{\text{internal}}$ consists of standard Grassmann multiplication, the nonlinear theories (3.44) to (3.60) describe a $N = 1$ supersymmetric extension of the multi-graviton gauge theories of Cutler and Wald, where the $n > 1$ spin-2 fields $h_{aBB'}{}^{\mu}$ are each formally commuting while the $n' > 1$ spin-3/2 fields $\psi_{aB}{}^{\Lambda}$ are each formally anticommuting (with all products obeying ordinary associativity). The odd-Grassmann multiplication rules in this extension can be

modified to intertwine with charge conjugation, so ${\psi_{aB}}^{\Lambda}$ is no longer strictly anticommuting while ${h_{aBB'}}^{\mu}$ remains commuting (and products are no longer strictly associative) analogously to the single graviton and graviton case ($n = n' = 1$) discussed in Ref. [21]. The field coupling constants can be generalized correspondingly, such that commutativity and associativity with respect to $\mathbb{A}^{\text{odd}}_{\text{coupling}}$ hold only up to complex conjugation.

A different extension of the Cutler-Wald multi-graviton theories is given by using standard multiplication rules for the spin-2 and spin-3/2 fields, corresponding to when $\mathbb{A}_{\text{internal}}$ is simply the algebra of real numbers, $\mathbb{R}$, while $\mathbb{A}_{\text{coupling}}$ is now a real, graded-commutative, associative algebra, whose structure constants ${a^{\mu}}_{\alpha\beta}$, ${b^{\mu}}_{\Omega\Gamma}$, ${c^{\Lambda}}_{\Gamma\alpha}$ are characterized by the relations

$$ {a^{\mu}}_{[\alpha\beta]} = 0, \tag{3.61} $$

$$ {b^{\mu}}_{(\Omega\Gamma)} = 0, \quad {c^{\Lambda}}_{(\Gamma|\nu|}{b^{\nu}}_{\Sigma)\Omega} = 0, \tag{3.62} $$

together with

$$ {a^{\mu}}_{\nu\alpha}{b^{\nu}}_{\Lambda\Gamma} = {b^{\mu}}_{\Lambda\Omega}{c^{\Omega}}_{\Gamma\alpha}, \quad {c^{\Lambda}}_{\Omega\alpha}{c^{\Omega}}_{\Gamma\beta} = {c^{\Lambda}}_{\Gamma\nu}{a^{\nu}}_{\alpha\beta}. \tag{3.63} $$

In this multi-graviton and multi-gravitino gauge theory, because ${h_{aBB'}}^{\mu}$ as well as ${\psi_{aB}}^{\Lambda}$ are each ordinary commuting fields, supersymmetry gauge invariance of the theory relies on the antisymmetry properties of the spin-3/2 coupling constants (3.62), which requires that the number n' of spin-3/2 fields be at least two. (This is somewhat analogous to aspects of the structure of Yang-Mills gauge theory for a set of spin-1 fields.) These coupling constants in the field-coupling algebra $\mathbb{A}_{\text{internal}}$, moreover, can be generalized to become complex valued with antisymmetry properties that now involve complex conjugation

$$ {b^{\mu}}_{\Lambda\Gamma} = -{\bar{b}^{\mu}}_{\Gamma\Lambda} \neq -{b^{\mu}}_{\Gamma\Lambda}, \quad {c^{\Lambda}}_{\Gamma\nu}{b^{\nu}}_{\Sigma\Omega} = {c^{\Lambda}}_{\Omega\nu}{\bar{b}^{\nu}}_{\Gamma\Sigma} \neq {c^{\Lambda}}_{\Omega\nu}{b^{\nu}}_{\Gamma\Sigma}. \tag{3.64} $$

The previous two types of multi-graviton and multi-gravitino theories (3.44) to (3.60) can be reformulated simply in terms of an $\mathbb{A}^{\text{even}}_{\text{coupling}}$-valued tetrad field and an $\mathbb{A}^{\text{odd}}_{\text{coupling}}$-valued vector-spinor field

$$ {\boldsymbol{e}_a}^{BB'} = \mathbb{1}{\sigma_a}^{BB'} + {\boldsymbol{h}_a}^{BB'}, \quad {\boldsymbol{\varphi}_a}^{B} = {\boldsymbol{\psi}_a}^{B} \tag{3.65} $$

through employing a unit element $\mathbb{1}^{\mu}$ in $\mathbb{A}_{\text{coupling}}$ (appending one if none exists [16]). This tetrad ${\boldsymbol{e}_a}^{BB'}$ is related to the spin-2 field variable (3.45) in a similar way to the metric tensor field (3.26) and (3.36) in the Cutler-Wald multi-graviton theories. An analogous relation is evident between the vector-spinor ${\boldsymbol{\varphi}_a}^{B}$ and spin-3/2 field variable (3.44).

A more interesting and highly novel multi-graviton and multi-gravitino gauge theory is obtained when $\mathbb{A}^{\text{even}}_{\text{internal}}$ consists of formal rules of anticommutative, anti-associative multiplication while $\mathbb{A}^{\text{odd}}_{\text{internal}}$ consists of standard commutative, associative multiplication rules. This makes the spin-2 fields ${h_{aBB'}}^{\mu}$ each formally anticommuting and the spin-3/2 fields ${\psi_{aB}}^{\Lambda}$ each formally commuting, as summarized by the rules (3.37) together with

$$\vartheta_1\vartheta_2 = \vartheta_2\vartheta_1, \quad \vartheta_1(\vartheta_2\vartheta_3) = (\vartheta_1\vartheta_2)\vartheta_3, \tag{3.66}$$

$$\vartheta_1(\varrho_1\varrho_2) = (\vartheta_1\varrho_1)\varrho_2, \quad (\vartheta_1\vartheta_2)\varrho_1 = \vartheta_1(\vartheta_2\varrho_1) = (\vartheta_1\varrho_1)\vartheta_2, \tag{3.67}$$

where ϑ's stand for ${\psi_{aB}}^{\Lambda}$ or odd-graded products of ${\psi_{aB}}^{\Lambda}$'s and ${h_{aBB'}}^{\mu}$'s, while ϱ's stand for ${h_{ab}}^{\mu}$ or even-graded products of ${\psi_{aB}}^{\Lambda}$'s and ${h_{aBB'}}^{\mu}$'s. These formal multiplication rules (3.37), (3.66) and (3.67) reverse the usual spin-statistics relation at the classical level, and therefore we will refer to them as reverse-Grassmann graded-associative multiplication rules. At the same time, the field-coupling algebra $\mathbb{A}_{\text{coupling}}$ is given by an arbitrary $n+n'$ dimensional anticommutative, graded-associative algebra, whose n dimensional even part $\mathbb{A}^{\text{even}}_{\text{coupling}}$ is anti-associative. The structure constants ${a^{\mu}}_{\alpha\beta}$, ${b^{\mu}}_{\Omega\Gamma}$, ${c^{\Lambda}}_{\Gamma\alpha}$ of $\mathbb{A}_{\text{coupling}}$ are characterized by the relations (3.38), (3.62) and (3.63). It is straightforward to show that the antisymmetry relations (3.38) and (3.62) imply

$${a^{\mu}}_{\nu\sigma}{b^{\nu}}_{\Lambda\Gamma}{b^{\sigma}}_{\Sigma\Omega} = 0. \tag{3.68}$$

For these relations to be satisfied, there is, in general, a tradeoff between the degree of nontriviality of ${b^{\mu}}_{\Omega\Gamma}$ compared with ${c^{\Lambda}}_{\Gamma\alpha}$ and ${a^{\mu}}_{\alpha\beta}$. We consider two simple examples. First, ${b^{1}}_{1\,2} \neq 0$, ${b^{2}}_{1\,3} \neq 0$, ${b^{3}}_{2\,3} \neq 0$, ${b^{4}}_{1\,4} \neq 0$, ${b^{5}}_{2\,4} \neq 0$, ${b^{6}}_{3\,4} \neq 0$, and ${c^{4}}_{1\,3} = -{c^{4}}_{2\,2} = {c^{4}}_{3\,1} \neq 0$, ${c^{5}}_{1\,5} = -{c^{5}}_{2\,4} = {c^{5}}_{4\,1} \neq 0$, ${c^{6}}_{2\,6} = -{c^{6}}_{3\,5} = {c^{6}}_{4\,3} \neq 0$, (all others zero, taking into account antisymmetry) with $n' = 8, n = 6$, which produces a fourteen-dimensional anticommutative, associative algebra where the entire even subalgebra is most nilpotent. (This example arises by forming all possible anticommutative products of four odd generators: $o_1, o_2, o_3, o_4, o_{12} \equiv o_1o_2, o_{13} \equiv o_1o_3, o_{23} \equiv o_2o_3, o_{14} \equiv o_1o_4, o_{24} \equiv o_2o_4, o_{34} \equiv o_3o_4, o_{123} \equiv o_1o_{23} = -o_2o_{13} = o_3o_{12}, o_{124} \equiv o_1o_{24} = -o_2o_{14} = o_4o_{12}, o_{134} \equiv o_1o_{34} = -o_3o_{14} = o_4o_{13}, o_{234} \equiv o_2o_{34} = -o_3o_{24} = o_4o_{23}$ such that $o_{12}o_{34} = o_{13}o_{24} = o_{23}o_{14} = 0$, and all other products equal to zero.) Second, ${a^{3}}_{1\,2} \neq 0$, and ${c^{1}}_{2\,1} \neq 0$, ${c^{3}}_{4\,1} \neq 0$, ${b^{2}}_{2\,4} \neq 0$, ${b^{3}}_{1\,4} = {b^{3}}_{2\,3} \neq 0$, (all others zero, taking into account (anti)symmetry) with $n = 3, n' = 4$, producing a seven-dimensional associative, anticommutative algebra. (This example arises by forming all possible anticommutative

even-graded products of a single even generator and four odd generators: u, o_1, o_2, o_3, o_4 such that $uo_2 = o_1, uo_3 = o_4, o_{14} \equiv o_1 o_4, o_{23} \equiv o_2 o_3, o_{24} \equiv o_2 o_4$ and hence $uo_{24} = o_{14} = -uo_{42} = o_{23}$, with all other products equal to zero.) It is easy to extend these two examples to obtain an anticommutative, graded-associative algebra by enlarging the even subalgebra to make it anti-associative (namely, via appending additional even generators).

Compared with the supersymmetric extensions of the Cutler-Wald multi-graviton theories, note that the new multi-graviton and multi-gravitino theories presented here do not have a formulation in terms of $\mathbb{A}_{\text{coupling}}$-valued fields (3.65), since no anticommutative algebra can admit a unit element.

Finally, a scalar Lagrangian for these multi-graviton and multi-gravitino gauge theories (3.44) to (3.60) is readily obtained by the same kind of procedure as summarized in Propositions 3 to 5 for the multi-graviton gauge theories (3.26) to (3.34).

Proposition 6. *(i) For $\mathbb{A}_{\text{coupling}}$ given by a commutative, associative graded algebra with a unit element, a scalar Lagrangian for the nonlinear theories (3.44) to (3.60) using formal Grassmann multiplication rules for ${h_{aBB'}}^{\mu}$ and ${\psi_{aB}}^{\Lambda}$ is obtained by*

$$L_{\text{spin2,3/2}} = {q^*}_{\mu} L^{\mu}_{\text{spin2,3/2}} \tag{3.69}$$

where ${q^*}_{\mu}$ projects onto a most nilpotent element (or, if none, a unit element) in $\mathbb{A}_{\text{coupling}}$.

(ii) For $\mathbb{A}_{\text{coupling}}$ given by an anticommutative, graded-associative algebra, a scalar Lagrangian for the nonlinear theories (3.26) to (3.34) using formal reverse-Grassmann graded-associative multiplication rules for ${h_{aBB'}}^{\mu}$ and ${\psi_{aB}}^{\Lambda}$ is constructed in terms of components $q_{\mu\nu} = q_{[\mu\nu]}$, ${q'}_{\Lambda\Gamma} = {q'}_{[\Lambda\Gamma]}$ of an invariant nondegenerate inner product on $\mathbb{A}_{\text{coupling}}$ by

$$L_{\text{spin2,3/2}} = q_{\mu\nu} L^{\mu\nu}_{\text{spin2}} + {q'}_{\Lambda\Gamma} L^{\Lambda\Gamma}_{\text{spin3/2}} \tag{3.70}$$

where $L^{\mu\nu}_{\text{spin2}}$ and $L^{\Lambda\Gamma}_{\text{spin3/2}}$ are the coefficients defined from $L^{\sigma}_{\text{spin2,3/2}} = {a^{\sigma}}_{\mu\nu} L^{\mu\nu}_{\text{spin2}} + {b^{\sigma}}_{\Lambda\Gamma} L^{\Lambda\Gamma}_{\text{spin3/2}} +$ linear ${h_{aBB'}}^{\sigma}$ terms.

(iii) For $\mathbb{A}_{\text{coupling}}$ given by a graded-commutative, associative algebra, a scalar Lagrangian for the nonlinear theories (3.26) to (3.34) using ordinary multiplication rules for ${h_{aBB'}}^{\mu}$ and ${\psi_{aB}}^{\Lambda}$ is obtained from either (3.69) if $\mathbb{A}_{\text{coupling}}$ possesses a unit element, or (3.70) if $\mathbb{A}_{\text{coupling}}$ is completely nilpotent.

In case (i), the most nilpotent ideal in $\mathbb{A}^{\text{even}}_{\text{coupling}}$ is required to be one-dimensional, which holds if $\mathbb{A}_{\text{coupling}}$ is any even-Grassmann graded algebra such that the number of generators

is an even integer. In case (ii), we note that the invariance property of the inner product is characterized by the relation (3.43) together with

$$q_{\nu\mu} b^{\mu}{}_{\Lambda\Gamma} = q'{}_{\Lambda\Omega} c^{\Omega}{}_{\Gamma\nu}. \tag{3.71}$$

Further details will not be pursued for this case other than to remark that, as in the situation for anticommutative, anti-associative algebras $\mathbb{A}_{\text{coupling}}$ discussed in Sec. III B, invariance and nondegeneracy of an inner product necessarily forces some restrictions on the most nilpotent ideals associated with even and odd multiplication in $\mathbb{A}_{\text{coupling}}$. Case (iii) obviously requires the same conditions on $\mathbb{A}_{\text{coupling}}$ as noted in cases (i) and (ii).

IV. CONCLUDING REMARKS

This paper has obtained a complete classification of the possibilities allowed for non-higher-derivative classical gauge theories of a nonlinearly coupled set of any number of spin-2 fields and spin-3/2 fields with general formal internal multiplication rules. The classification includes, as a special case, all allowed possibilities for a nonlinearly coupled set of any number of spin-2 fields alone. This classification in the spin-2 case has found a novel type of nonlinear gauge theory for two or more spin-2 fields, with coupling constants given by anti-commutative, anti-associative algebras, and using formal multiplication rules that make the spin-2 fields anticommuting (while products obey anti-associativity). Moreover, apart from the previously known type of nonlinear gauge theories found by Cutler and Wald for one or more ordinary commuting spin-2 fields whose coupling constants arise from commutative, associative algebras, the classification shows that there are no other possibilities. Supersymmetric extensions of these nonlinear theories have been obtained and further proved to be the only allowed possibilities in the more general case when spin-3/2 fields are included.

The framework used for obtaining these main results consists of a deformation analysis of the linear gauge theory for a set of uncoupled spin-2 and spin-3/2 fields with a common internal vector space structure for each field as necessary for accommodating formal multiplication rules. Determining equations for deformation terms have been formulated and solved by the methods of Ref. [18, 19] generalized to the case of more than one spin-2 field and more than one spin-3/2 field. It is important to emphasize that no assumptions or special conditions were imposed here on the form considered for possible deformation terms

except for requiring these terms to involve no higher derivatives than those appearing in the linear field equations and abelian gauge symmetries.

The results and framework here give a culmination of much previous work in the literature on non-higher-derivative deformations for gauge theories of spin-2 fields [5, 6, 7, 8, 9, 10, 11, 15, 18, 26], including couplings to spin-3/2 fields [14, 31, 32, 33], in four spacetime dimensions. In particular, uniqueness results for such deformations are strengthened in this paper through the use of a nonsymmetric tensor analogous to a tetrad for the spin-2 field variable, which has given a strong no-go theorem on possibilities for deforming the local Lorentz symmetry on the spin-2 field variable (and spin-3/2 field variable), and for producing couplings that involve the skew part of the spin-2 field variable. At the same time, this paper considerably generalizes all previous frameworks by encompassing the most general possibilities for formal rules other than commutative, associative multiplication of spin-2 fields, and its extension to Grassmann multiplication of spin-3/2 fields, which has uncovered the novel existence of consistent nonlinear couplings for more than one anticommuting spin-2 field and supersymmetric extensions with more than one commuting spin-3/2 field.

Some further lines of work are suggested by the main results discussed above. Do the new multi-graviton and multi-gravitino gauge theories found here have a well-posed initial value formulation as classical field theories? This could be expected to hold by comparison with the initial value formulation of N= 1 supergravity as a single graviton and gravitino classical field theory. Is there a consistent quantization for the new gauge theories with the multi-graviton and multi-gravitino quantum fields obeying reversed spin-statistics commutation relations that mirror the formal multiplication rules found here at the classical level? This could be investigated first for a model nonlinear theory of, for example, coupled anticommuting scalar fields and commuting neutrino or Dirac fields as obtained in a manner analogous to way in which the new multi graviton and gravitino gauge theories are derived from the formulation of N= 1 supergravity for an algebra-valued spin-2 field and spin-3/2 field.

APPENDIX: GRADED ALGEBRAS

In this appendix, some definitions and structure of graded algebras are summarized. Throughout let $\mathbb{A}$ be an algebra possessing a grading that determines a decomposition into even and odd subspaces $\mathbb{A} = \mathbb{A}^{\rm even} \oplus \mathbb{A}^{\rm odd}$ such that products of two even elements are

even, products of two odd elements are even, and products of one even element and one odd element are odd.

A unit element $\mathbb{1}$, if one exists, is an element in $\mathbb{A}^{\text{even}}$ such that $\mathbb{1}x = x\mathbb{1} = x$ for all elements x in $\mathbb{A}$. A non-unit element element x in $\mathbb{A}$ is nilpotent if the p-fold product of itself, denoted x^p, vanishes for some integer $p > 1$. A most nilpotent element $\mathbb{A}$ is a nilpotent element y whose product with all other nilpotent elements x vanishes, $xy = 0$. $\mathbb{A}$ is called completely nilpotent if every element in $\mathbb{A}$ is nilpotent.

$\mathbb{A}$ is an associative algebra if all multiplication obeys associativity, namely, $x(yz) = (xy)z$ for all elements x, y, z in $\mathbb{A}$. An associative algebra $\mathbb{A}$ is (anti) commutative if all multiplication is (anti) commutative, namely, $xy = (-)yx$ for all elements x, y in $\mathbb{A}$. An associative algebra $\mathbb{A}$ is graded-commutative if odd multiplication is anticommutative, even multiplication is commutative, and even-odd multiplication is commutative, namely, $xy = yx$ for all elements x, y in $\mathbb{A}^{\text{even}}$, $xy = -yx$ for all elements x, y in $\mathbb{A}^{\text{odd}}$, and $xy = yx$ for all elements x in $\mathbb{A}^{\text{even}}$ and y in $\mathbb{A}^{\text{odd}}$.

A nilpotent Grassmann algebra $\mathbb{G}_{\text{nilpotent}}$ is an associative, graded-commutative algebra $\mathbb{A}$ formed by the span of all possible products of some number of odd generators, namely, $o_1, o_2, \ldots, o_{12} \equiv o_1 o_2, o_{13} \equiv o_1 o_3, o_{23} \equiv o_2 o_3, \ldots, o_{123} \equiv o_1 o_2 o_3, \ldots$, etc. A Grassmann algebra $\mathbb{G}_{\text{unit}}$ is the enlargement of a nilpotent Grassmann algebra obtained by appending a unit element $\mathbb{1}$. A (nilpotent) even-Grassmann algebra is an associative, commutative algebra isomorphic to the even subalgebra $\mathbb{G}_{\text{even}}$ of a (nilpotent) Grassmann algebra. Equivalently, a (non-nilpotent) even-Grassmann algebra is formed by the span of (a unit element together with) all possible products of some number of even generators whose square is zero, namely, $u_1, u_2, u_3, \ldots, u_{12} \equiv u_1 u_2, u_{13} \equiv u_1 u_3, u_{23} \equiv u_2 u_3, \ldots, u_{123} \equiv u_1 u_2 u_3, \ldots$, etc.

$\mathbb{A}$ is an anticommutative, graded-associative algebra if products of two elements are anticommutative, and products of three or more even elements are anti-associative, while all other products are associative, namely, $xy = -yx$, $x(yz) = -(xy)z$ for all elements x, y, z in $\mathbb{A}^{\text{even}}$, and $x(yz) = (xy)z$ for all elements where at least one of x, y, z is in $\mathbb{A}^{\text{odd}}$.

A reverse-Grassmann graded-associative algebra is defined here to be an algebra $\mathbb{A}$ such that even multiplication is anticommutative and anti-associative, odd multiplication and even-odd multiplication are each commutative and associative.

The set of most nilpotent elements in an algebra $\mathbb{A}$ forms an ideal, while the set of nilpotent elements also forms an ideal if multiplication is either (anti) associative or graded-

associative, and if multiplication is either (anti) commutative or graded-commutative.

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